INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided for each problem. This is very important! Even if you are using LaTeX, make sure your solutions fit into the given space.

2. Make sure you write your name and NetID in the space provided above.

3. After you are done writing, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances! The submitted PDF should have all the seven pages in the correct order.

4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.
Part I

Problem 1. [10 pts]
How many integer solutions are there to the following system of equations?

\[ x \times y \times z = 256 \]

\[ x, y, z \geq 2. \]

Justify your answer. You can leave your answer in terms of factorials and/or binomial coefficients.

Proof. Let \( x = 2^X \), \( y = 2^Y \), and \( z = 2^Z \). Then, the above equation becomes \( 2^{X+Y+Z} = 256 = 2^8 \), and thus, \( X + Y + Z = 8 \). The constraints becomes \( X \geq 1, Y \geq 1, Z \geq 1 \). This is the pirates and gold bars setup: we want to distribute 8 identical goldbards among 3 pirates so that every pirate gets at least one goldbar. First, we satisfy their minimum requirement and give one bar to every pirate. That leaves us with 5 bars to distribute among 3 pirates, which is the same as the number of binary strings with 5 zeros and 2 ones (2 ones will partition the zeros into three parts), and thus the answer is \( \binom{5+2}{2} \). \( \square \)
Problem 2. [10 pts]
Let $A$ and $B$ be finite sets such that $|A| = 15$ and $|B| = 5$. How many 3-to-1 functions with $A$ as domain and $B$ as codomain are there? Justify your answer. You can leave your answer in terms of factorials and/or binomial coefficients.

Proof. Let $B = \{b_1, b_2, \ldots, b_5\}$. In any 3-to-1 function from $A$ to $B$, 3 elements of $A$ must map to $b_1$, 3 elements should map to $b_2$, ..., 3 elements must map to $b_5$. Also, since we are looking at functions from $A$ to $B$, every element of $A$ maps to exactly one element of $B$. To specify a 3-to-1 function, all we need to specify is which three elements of $A$ map to $b_1$, then from the remaining 12 elements (after having specified which elements go to $b_1$), we need to specify which three go to $b_2$, and so on. The number of ways of choosing 3 elements from 15 elements that map to $b_1$ is $\binom{15}{3}$, the number of ways of choosing 3 elements out of 12 that go to $b_2$ is $\binom{12}{3}$, number of ways of choosing 3 out of 9 elements to map to $b_3$ is $\binom{9}{3}$, the number of elements of choosing three elements to map to $b_4$ is $\binom{6}{3}$, and finally there are $\binom{3}{3}$ ways of choosing three elements to map $b_5$ after we have fixed the choices for $b_1, \ldots, b_4$. Thus, the total number of ways using the generalized product rule is:

$$\binom{15}{3} \binom{12}{3} \binom{9}{3} \binom{6}{3} \binom{3}{3} = \frac{15!}{(3!)^5}.$$

$\square$
Part II

Problem 3. [20 pts]
A robot is located at the coordinates \((p, q, r)\) on a 3D maze. The robot needs to go to the origin, i.e. the point \((0, 0, 0)\). From any location \((x, y, z)\) the valid moves for the robot are one of the following:

- \((x, y, z) \rightarrow (x - 1, y, z)\) if \(x > 0\).
- \((x, y, z) \rightarrow (x, y - 1, z)\) if \(y > 0\).
- \((x, y, z) \rightarrow (x, y, z - 1)\) if \(z > 0\).

Count the number of different valid paths that the robot can take to go from \((p, q, r)\) to \((0, 0, 0)\). A valid path is a path in which all the intermediate moves are valid. For example, for \(p = q = r = 2\), a valid path from \((p, q, r)\) to \((0, 0, 0)\) is the following:

\[
(2, 2, 2) \rightarrow (2, 1, 2) \rightarrow (2, 1, 1) \rightarrow (1, 1, 1) \rightarrow (0, 1, 1) \rightarrow (0, 1, 0) \rightarrow (0, 0, 0).
\]

You must justify your answer. Write you answer in terms of factorials and/or binomial coefficients. Also, you have this page and the next page for writing the solution.

**Proof.** Any path valid path from \((p, q, r)\) to \((0, 0, 0)\) must make \(p\) valid moves in the \(x\) direction, \(q\) valid moves in the \(y\) direction, and \(r\) valid moves in the \(z\) direction. We can represent a path from \((p, q, r)\) to \((0, 0, 0)\) by a sequence of length \(p + q + r\) made from \(p\) copies of the letter “\(x\)”, \(q\) copies of “\(y\)”, and \(r\) copies of “\(z\)”, where “\(x\)” means “make a valid move in the \(x\) direction”, “\(y\)” means “make a valid move in the \(y\) direction”, and “\(z\)” means “make a valid move in the \(z\) direction”. Conversely, every sequence of length \(p + q + r\) which only contains \(p\) “\(x\)”’s, \(q\) “\(y\)”’s, and \(r\) “\(z\)”’s can be interpreted as a path from \((p, q, r)\) to \((0, 0, 0)\). Thus by the bijection method the number of valid paths is the same as the number of sequences of that type.

The number of sequence of length \(p + q + r\) that contain \(p\) \(x\)’s, \(q\) \(y\)’s, and \(r\) \(z\)’s, using the sequences with repetitions formula, is

\[
\frac{(p + q + r)!}{p!q!r!}.
\]
More space for problem 3 solution:
Problem 4. [20 pts]
A driver and their racing team are taking part in a race that involves doing 80 laps around a circuit. They have 10 distinct sets of tires (say $T_1, T_2, \ldots, T_{10}$) and want to use each set of tires for at least one lap during the race. There are, however, some constraints:

- The tires cannot be changed during a lap, and can only be changed at the beginning or end of a lap. Obviously, the end of lap 1 is the same as the beginning of lap 2, and so on, and so there are exactly 80 opportunities for the team to possibly change the tires (It doesn’t make sense to change tires after the last lap, i.e. after the race has ended).

- Once a set of tires is removed from the car, they can never be reused and must be discarded. For example, if at the beginning of lap 1, i.e., the beginning of the race, the team installs the set of tires $T_5$, and lets the driver race with those tires for 10 laps, and replaces the set $T_5$ with the set $T_1$ at the end of lap 10, then they can never re-use the set $T_5$ again for the rest of the race – that set of tires must be discarded.

A tire change schedule is a schedule that instructs the team when and how they should change tires, respecting the above constraints. For example, a valid tire change schedule is as follows: Use set $T_1$ for the first 20 laps, then use set $T_5$ for the next 10 laps, then use set $T_3$ for the next 15 laps, then use set $T_2$ for the next 15 laps, and use set $T_4$ for the last 20 laps.

Find the total number of possible tire change schedules that respect all the constraints. Justify your answer. You may leave your answer in terms of factorials and/or binomial coefficients. You have this and the next page for writing your solution.

Proof. Let’s divide the 80 laps into 10 phases, where each phase lasts at least 1 lap and uses a distinct (distinct from other phases) set of tires. So we have Phases 1 – 10. Let’s not decide yet which tires to use in what phase — we will do that at the very end. First we will decide how long should every phase last. Let $x_i$, for $1 \leq i \leq 10$, denote the number of laps that Phase $i$ lasts for. We know that

$$x_1 + \ldots + x_{10} = 80,$$

with the conditions that $x_1, x_2, \ldots, x_{10} \geq 1$ and all of them must be integers (a phase cannot contain half a lap — that would be mean we have to change the tires in the middle of a lap which is not allowed). We want to find the number of solutions to this equation and that will give us the number of ways to divide the 80 laps between the 10 phases. Let’s first satisfy the minimum requirements, and give 1 to every variable, and thus we will be left with 70. We now have to distribute 70 among 10 variables and thus the number of ways of doing that is the number of binary strings with 9 ones (and thus 10 partitions) and 70 zeros, which is $\binom{79}{9}$.

So far we have found the number of ways in which we can partition the 80 laps into ten phases of at least one lap. Having fixed a choice for the duration of the phases, we must specify which set of tires is to be used when, and there are $10!$ ways of deciding the order.
Notice that the partitioning of the laps into phases is *independent* from the order in which we use the tires, and thus using the product rule we have that the total number of tire change schedules is \((\binom{79}{9}) \cdot 10!\).
More space for problem 4 solution: