Recitation sample problems

David Menendez (davemm@cs.rutgers.edu)

Some worked examples I wrote up for past recitations.

1 Mathematical Induction

Unfortunately, all the examples I’ve done in class are from the book, and we’re not supposed to post answers on-line. So here’s a proof by induction of a well-known theorem:

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \]

**Proof:** By induction.

**Basis step:** We need to show that \( 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \).

**Inductive step:** Assume that \( 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \) for an arbitrary integer \( k \geq 1 \).\(^1\) We need to show that \( 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2} \).

\[
1 + 2 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) \quad \text{by our assumption}
\]

\[
= \frac{k(k + 1) + 2(k + 1)}{2}
\]

\[
= \frac{(k + 1)(k + 2)}{2}
\]

This completes the inductive step.

Thus, we conclude that \( 1 + \cdots + n = \frac{n(n+1)}{2} \) for all integers \( n \geq 1 \).

1.1 Strong induction

Let’s say we need to write a simple program to raise an integer to a power. That is, for some \( m \) and \( n \), we need to calculate \( m^n \). The easiest way to do this is to multiply \( m \) by itself \( n - 1 \) times,

\(^1\)This is the inductive hypothesis.
so that $m^4 = m \cdot m \cdot m \cdot m$. I claim we can use fewer multiplications by setting $m^2 = m \cdot m$ and $m^4 = m^2 \cdot m^2$ and so forth.

In general,

$$m^n = \begin{cases} 
  m^k \cdot m^k & \text{if } n = 2k \\
  m^k \cdot m^k \cdot m & \text{if } n = 2k + 1 
\end{cases}$$

So calculating $m^n$ requires calculating $m^{\lfloor n/2 \rfloor}$ and then performing 1 or 2 additional multiplications.

Let’s use strong induction to prove that this method requires fewer than $n - 1$ multiplications to calculate $m^n$ for $n \geq 4$.

**Hypothesis**

For all integers $n \geq 4$, $m^n$ can be calculated with fewer than $n - 1$ multiplications.

**Basis step:** To find $m^4$, we first calculate $m^2 = m \cdot m$ and then $m^4 = m^2 \cdot m^2$. This requires 2 multiplications, which is fewer than $4 - 1 = 3$.

**Inductive step:** Let $k$ be an arbitrary integer greater than or equal to 4. Assume that we can calculate $m^i$ in fewer than $i$ multiplications for all $i$ such that $4 \leq i \leq k$. We need to show that we can calculate $m^{k+1}$ with fewer than $k$ multiplications. Because our method involves calculating $m^{\lfloor (k+1)/2 \rfloor}$, we will need to consider whether $k + 1$ is odd or even, and separately consider the cases where $\lfloor (k + 1)/2 \rfloor < 4$:

1. **Case 1:** $k = 4$. We will calculate $m^5 = m^2 \cdot m^2 \cdot m$. Since finding $m^2$ requires 1 multiplication, we will need 3 to calculate $m^5$.

2. **Case 2:** $k = 5$. We will calculate $m^6 = m^3 \cdot m^3$. Since finding $m^3$ requires 2 multiplications, we will need 3 to calculate $m^6$.

3. **Case 3:** $k = 6$. We will calculate $m^7 = m^3 \cdot m^3 \cdot m$. Since finding $m^3$ requires 2 multiplications, we will need 4 to calculate $m^7$.

4. **Case 4:** $k \geq 7$. In this case, $\lfloor (k + 1)/2 \rfloor \geq 4$, so we may use the inductive hypothesis. We will consider two sub-cases, where $k + 1$ is even or odd:

   a. **Case 4.1:** $k + 1$ is even. Since $k + 1$ is even, there must be an integer $j$ such that $k + 1 = 2j$. Therefore, we will calculate $m^{k+1} = m^j \cdot m^j$. By the inductive hypothesis, we can calculate $m^j$ in fewer than $j - 1$ multiplications. Therefore, we need fewer than $j$ multiplications to calculate $m^{k+1}$.

   Since $k \geq 7$, $j < k$ and therefore $m^{k+1}$ can be calculated in fewer than $k$ multiplications.

   b. **Case 4.2:** $k + 1$ is odd. Since $k + 1$ is odd, there must be an integer $j$ such that $k + 1 = 2j + 1$. Therefore, we will calculate $m^{k+1} = m^j \cdot m^j \cdot m$. By the inductive hypothesis, we can calculate $m^j$ in fewer than $j - 1$ multiplications. Therefore, we need fewer than $j + 1$ multiplications to calculate $m^{k+1}$.

---

2This is not the inductive hypothesis.
Since $k \geq 7$, $j + 1 < k$, and therefore $m^{k+1}$ can be calculated in fewer than $k$ multiplications.

All the cases show that $m^{k+1}$ can be calculated in fewer than $k$ multiplications. This completes the inductive step.

Thus, by strong induction, $m^n$ can be calculated with fewer than $n - 1$ multiplications for all integers $n \geq 4$.

2 Natural Deduction

2.1 One problem three ways

Normally, if you needed to show something like $\neg(P \lor Q) \rightarrow (\neg P \land \neg Q)$, you would simply note that this is an obvious consequence of DeMorgan’s laws and that would be sufficient. But since we’re learning how to prove things, here are two less-implicit proofs.

First, using logical equivalences:

\[
\neg(P \lor Q) \rightarrow (\neg P \land \neg Q) \equiv (\neg P \land \neg Q) \rightarrow (\neg P \land \neg Q) \quad \text{DeMorgan’s Laws}
\]
\[
\equiv (\neg P \land \neg Q) \lor (\neg P \land \neg Q) \quad \rightarrow \text{Equivalence}
\]
\[
\equiv \text{True} \quad \neg \text{Negation}
\]

Here, we used DeMorgan’s laws to replace $\neg(P \lor Q)$ with $\neg P \land \neg Q$, then we rewrote the implication into a disjunction, which was trivially true because one term was the negation of the other.

Next, using natural deduction.

<table>
<thead>
<tr>
<th></th>
<th>( \neg(P \lor Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neg(P \lor Q) )</td>
</tr>
<tr>
<td>2</td>
<td>( P )</td>
</tr>
<tr>
<td>3</td>
<td>( P \lor Q \lor I \ 2 )</td>
</tr>
<tr>
<td>4</td>
<td>( \text{False} \lor I \ 1,3 )</td>
</tr>
<tr>
<td>5</td>
<td>( \neg P \lor I \ 2,4 )</td>
</tr>
<tr>
<td>6</td>
<td>( Q \lor I \ 6 )</td>
</tr>
<tr>
<td>7</td>
<td>( P \lor Q \lor I \ 6 )</td>
</tr>
<tr>
<td>8</td>
<td>( \text{False} \lor I \ 1,7 )</td>
</tr>
<tr>
<td>9</td>
<td>( \neg Q \lor I \ 6,8 )</td>
</tr>
<tr>
<td>10</td>
<td>( \neg P \land \neg Q \lor I \ 5,9 )</td>
</tr>
<tr>
<td>11</td>
<td>( \neg(P \lor Q) \rightarrow (\neg P \land \neg Q) \lor I \ 1,10 )</td>
</tr>
</tbody>
</table>
We want to show an implication, so we assume the premise (1) and attempt to show the consequence (10). Since the consequence is a conjunction, we need to show both parts (5,9). These are negations, so we can prove them by assuming the opposite (2,6) and showing a contradiction (3 and 7 contradict 1).

2.2 Some more examples

\((\neg P \lor \neg Q) \rightarrow \neg(P \land Q)\):

\[
\begin{array}{c|l}
1 & \neg P \lor \neg Q \\
2 & P \land Q \\
3 & \neg P \\
4 & P & \land E 2 \\
5 & \text{False} & \text{False Intro 3,4} \\
6 & \neg Q \\
7 & Q & \land E 2 \\
8 & \text{False} & \text{False Intro 6,7} \\
9 & \text{False} & \lor E 1,5,8 \\
10 & \neg(P \land Q) & \neg I 2,9 \\
11 & (\neg P \lor \neg Q) \rightarrow \neg(P \land Q) & \rightarrow I 1,10 \\
\end{array}
\]

Here, since we are trying to conclude something from a disjunction, we have to use or-elimination by showing that we can reach the same conclusion (in this case a contradiction) from both terms.
$((P \lor Q) \land R) \rightarrow ((P \land R) \lor (Q \land R))$:

1. $(P \lor Q) \land R$
2. $R$ \hspace{1cm} \land E 1
3. $P \lor Q$ \hspace{1cm} \land E 1
4. $P$
5. $P \land R$ \hspace{1cm} \land I 2,4
6. $(P \land R) \lor (P \land Q)$ \hspace{1cm} \lor I 5
7. $Q$
8. $Q \land R$ \hspace{1cm} \land I 2,7
9. $(P \land R) \lor (Q \land R)$ \hspace{1cm} \lor I 8
10. $(P \land R) \lor (Q \land R)$ \hspace{1cm} \lor E 3,6,9
11. $((P \lor Q) \land R) \rightarrow ((P \land R) \lor (Q \land R))$ \hspace{1cm} \rightarrow I 1,11

$((P \land Q) \lor R) \rightarrow ((P \lor R) \land (Q \lor R))$:

1. $(P \land Q) \lor R$
2. $P \land Q$
3. $P$ \hspace{1cm} \land E 2
4. $P \lor R$ \hspace{1cm} \lor I 3
5. $Q$ \hspace{1cm} \land E 2
6. $Q \lor R$ \hspace{1cm} \lor I 5
7. $(P \lor R) \land (Q \lor R)$ \hspace{1cm} \lor I 4,6
8. $R$
9. $P \lor R$ \hspace{1cm} \lor I 8
10. $Q \lor R$ \hspace{1cm} \lor I 8
11. $(P \lor R) \land (Q \lor R)$ \hspace{1cm} \land I 9,10
12. $(P \lor R)\land (Q \lor R)$ \hspace{1cm} \lor E 1,7,11
13. $((P \land Q) \lor R) \rightarrow ((P \lor R) \land (Q \lor R))$ \hspace{1cm} \rightarrow I 1,12
$P \rightarrow (Q \lor R), (P \land Q) \rightarrow R \vdash P \rightarrow R$:

1. $P \rightarrow (Q \lor R)$  Premise
2. $(P \land Q) \rightarrow R$  Premise
3. $P$
4. $Q \lor R$  $\rightarrow$ E 1,3
5. $Q$
6. $P \land Q$  $\land$ I 3,5
7. $R$  $\rightarrow$ E 2,6
8. $R$
9. $R$  It 8
10. $R$  $\lor$ E 4,7,9
11. $P \rightarrow R$  $\rightarrow$ I 3,10