1 Solutions to HW2

Solution 1.1

We assume you know algorithm for finding \( k \)-th smallest number in array of \( n \) distinct numbers which runs in \( O(n) \) time. We first find median (it is \( n/2 \)-th smallest number, obviously depends on evenity or oddity of \( n \), lets ignore it here). Let median’s value be \( a \) lets make transformation of all other numbers as follows, number \( k \) is mapped to number \(|k - a|\) but each element still remember its original value. Now we simply find \((k + 1)\)-th smallest number and partition array on numbers smaller and numbers bigger. The \( k \) numbers closest to median are the smaller once, we just have to recover original values and that already conclude the solution. All operations take \( O(n) \) time and we just did constant time of them.

Solution 1.1

An alternative interpretation of Q1. Find \( \lfloor \frac{n-k}{2} \rfloor \)-th smallest element \( a \) and \( \lceil \frac{n+k}{2} \rceil \)-th smallest element \( b \) using order statistics, we can do it in \( O(n) \). Then we assume all the elements \( x \) for which \( a \leq x \leq b \), such elements we can also find in \( O(n) \). That already gives us a solution.

Solution 1.2

No, such operations are not ”commutative” and the following Figure 1 shows that.

![Figure 1: In the first series we remove first \( y \) and after \( x \), in the second opposite.](image)

Solution 1.3

Let call \( i \)-th element \( x \). Assume for contradiction we cannot decide all the smallest \( i - 1 \) elements and all the biggest \( n - i \) after series of comparisons defining the \( i \)-th element. Let \( R \) be comparison relation on input elements. We
add relations (edges) into $R$ from transitivity, see Fig. 2. Now if there is relation from $a$ to $b$ and relation from $b$ to $c$ in $R$ then there is also relation $a$ to $c$. We define set $S$ as all elements $s$ such that there is relation $s$ to $x$ (in $R$) and set $B$ as set of elements $b$ such that there is relation $x$ to $b$ (in $R$). If $S$ and $B$ and $x$ forms all elements we get to contradiction, because we can find all $i - 1$ smallest and $n - i$ biggest elements. So assume there is set $U$ of elements which are not in $S$ or $B$ (or not $x$ itself). There is the ordering $o$ of elements of $U$ (we might not know it, but it has to be there they are numbers), we add all relations (edges) from $o$ to relation $R$. Now we prove that all elements of $U$ can be smaller or bigger then $i$-th element and it only depends on future comparisons, see Fig. 3.

Suppose all the elements of $U$ are smaller then $x$ but bigger then all elements in $S$ (all in $R$) and suppose that it violates some previous relation, how is it possible? Either some element $s \in S$ is bigger then some element $u \in U$, but it is not possible because then $u$ would not be in $U$ but in $S$ or there is some element $b \in B$ which is smaller then some $v \in U$, but again that is not possible. Analogically we can treat case when $U$ are between $x$ and elements of $B$. That proved that there are at least two possible ordering and so $i$-th element cannot be determined.

**Exercise 1.4**

Because of problems with statement of Q4, we decided not to grade it. We apologize for this inconvenience.

**Solution 1.5**

If we insert new vertex to RB tree we run first function which insert red vertex $z$ into correct position and after we run fix-RB-tree function on $z$. You can notice that if there are no two consecutive red vertices fix-RB-tree function do nothing, so we are done, there is one red vertex. Assume there are two red consecutive vertices. Now it is enough to notice that in each operation of fix-RB-tree keep at
least one vertex red. Only exception is if we change color of root, but there are only two ways how it can be made. We either made rotation and change root element, but after we still have 2 red vertices so even we recolor root to black there is still one more. Other case is if we change color of root by recoloring, in this case is enough to notice that then at least one grandchild is still red.

**Solution 1.6**

To answer the question it is enough to show counterexample, see Fig. 4

![Diagram](image)

Figure 4: Following operations are counterexample to the problem 5, we insert $C$ and assume that $A < B < C$. 