1 Solutions to HW4

Solution 1.1
After CASCADING-CUT the situation looks as on Fig. 3 The potential before is:
\[ t + 2m = 3 + 2 \times 7 = 17 \]
The potential after operation is:
\[ t + 2m = 7 + 2 \times 5 = 17 \]
where \( t \) is number of trees and \( m \) is number of marked nodes.

Solution 1.2
Only situation when root of some heap in fibonacci heap is marked, is only if we call EXTRACT-MIN and a child \( c \) of minimum vertex is mark, and we put \( c \) and its subtree into heap list.

Solution 1.3
We first prove ”\( \Rightarrow \)”. We prove it by induction on the shortest path between vertices. First if there is path of length 0 then it is only single vertex, and so claim follows. Assume two vertices \( u \) and \( v \), so that \( d(u, v) = k \). Assume a shortest path \( p = (u = v_0, v_1, \ldots, v_{k-1}, v_k = v) \) from \( u \) to \( v \). From induction hypothesis \( u \) and \( v_{k-1} \) are in the same set, see Fig.1. Assume now that \( v \) and \( u \) are not in the same set. Because sets are disjoint, it implies that \( v _ {k-1} \) and \( v_k \) are not in the same set. But when algorithm using edge \( v_{k-1}v \) it makes \( v \) and \( v_{k-1} \) belongs to same edge, which is contradiction.

![Figure 1: A visualization a path \( p \) in the proof.](image_url)

Now we prove the opposite implication ”\( \Leftarrow \)”. We prove it again by induction but now the induction is on size of a set. First if set \( S \) has size 1 it immediately implies that there is path from every vertex to every vertex (there is just one). We assume that for all sets of size smaller than \( k \) the claim follows. Let size of \( S \) is \( k \), let assume that the last operation before \( S \) was created was union along edge \( e = uv \) of \( S_1 \) and \( S_2 \), but both \( S_1 \) and \( S_2 \) has size at most 1, so we also know \( S_1 \) and \( S_2 \) has size at most \( k - 1 \), see Fig.2. Great, we can use
induction, all vertices within $S_1$ are connected by path and same for $S_2$. Now assume arbitrary vertex $a$ in $S_1$ and arbitrary vertex $b$ in $S_2$, from induction assumption there is path from $a$ to $u$ and there is path from $v$ to $b$, that already implies there is path from $a$ to $b$ in $S$.

Solution 1.4
Let us assume that $n = 2^k$ for some $k$, we make the first $n$ operations to be MAKE-SET. Next we choose $n/2$ disjoint tuples of sets (all of them are of size 1), and do UNION on tuples. Now all of our sets are of size 2. Next we choose $n/4$ tuples of sets (all has size 2) and do UNION of tuples. We continue this way so in the end we have just one set, notice that in each step resulting set has rank +1. It is easy to see that every element is involved in $\log n$ UNION operations, which as result means that tree has height $\log n$. Let $x$ is element in the tree furthest from the root. Therefor path from $x$ to root is $\log n$ long. The rest of the operations are FIND-SET($x$). Now what is time complexity? The sequence of $n$ MAKE-SET operations takes $\Omega(n)$, there is $n - 1$ operations UNION and each takes $\Omega(1)$, so sequence of $n - 1$ operations UNION takes $\Omega(n)$. There is left $m - (n - 1 + n) = m - 2n - 1$ operations of FIND-SET($x$), each of them takes $\Omega(\log n)$. All together it is $\Omega(n) + \Omega(n) + \Omega((m - n) \log n)$, if $m$ is much larger than $n$ then the algorithm complexity is $\Omega(m \log n)$.

Solution 1.5
Figure 3: Stages of CASCADING-CUT. The last row is the resulting fibonacci heap.