Topics in Game Theory
Spring 2009
Lecture 1: Game theory, basic concepts, different kinds of equilibrium, etc. Price of anarchy results for networks

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1 Overview

In this class we define and explore the basic concepts of Game theory like: AGENTS, MECHANISM DESIGN, EQUILIBRIUM POINTS, DIFFERENT CONEPTS OF EQUILIBRIUM , PRICE OF ANACHY, PRICE OF STABILITY, NASH THEOREM AND EQUILIBRIUM . We then go on to examine a few example of games like: PRISONER'S DILLEMMA, BATTLE OF SEXES , MATCHING PENNY , PIGON'S EXAMPLE, BRASS' PARADOX that demonstrate these ideas.

2 Basic Concepts

We characterize an AGENT as someone selfish- trying to maximize his/her own benefits and minimize some objective function.MECHANISM DESIGN is the study of designing rules of a game or system to achieve a specific outcome, even though each agent may be self-interested [1].So the aim of MECHANISM DESIGN is to design rules which optimize social objective while individuals are trying to optimize their own objectives.
An EQUILIBRIUM POINT or simply EQUILIBRIUM is a state in which no person involved in the queue wants any change.More precisely, an EQUILIBRIUM is a state of the world where (economic) forces are balanced and in the absence of external influence the (equilibrium) values of economic variables will not change.

[1] wikipedia

So, somehow, everyone is comfortable in EQUILIBRIUM. The concept of Equilibrium is not unique. We will mainly consider Nash Equilibrium and Market Equilibrium. Again, Nash Equilibrium may be Pure or Mixed.
PRICE OF ANARCHY is the ratio between the worst possible equilibrium to the social optimal . So, Price of Anarchy is a measure of how well people do when they act selfishly as opposed to being co-ordinated by some central authority. On the other hand PRICE OF STABILITY is the ratio between the best possible equilibrium to the be value of the social optimal .

$$\mathsf{PriceofAnarchy} = \frac{\mathsf{worstEquilibrium}}{\mathsf{SocialOptimal}} \tag{1}$$

$$\mathsf{PriceofStability} = \frac{\mathsf{bestEquilibrium}}{\mathsf{SocialOptimal}}^{2} \tag{2}$$

3 Equilibrium

So an EQUILIBRIUM is a state where everyone is satisfied with the state of the affairs and no one wants change- assuming that there are no outside forces. Two main concepts of Equilibrium are:

1. NASH EQUILIBRIUM
2. MARKET EQUILIBRIUM

But equilibrium does not always exist for all problems. And the existance of equilibrium is a field of study. The question may be: 'Why do we care to study equilibrium? Is it always desirable to have equilibrium?'One way to explain an Equilibrium is that in Equilibrium, the system works efficiently and the social benifit is maximized. Equilibrium also gives an agent what its strategy should be. Many agents (e.g. Government) may be interested to impose restrictions on the game (by Mechanism Design) so that it might reach a target (e.g. stability) and the study of Equilibrium may help this.
The study of Equilibrium may also help to measure the performance of the current system to the optimal one to an approximate factor. This tells you how bad you are compared to the optimal.
Computer Scientists (as opposed to Economists) are more interested in the Computational complexity of Equilibrium. They might not care about equilibrium but they are interested about the dynamics of the game and the best possible responses to a situation.

4 Nash Equilibrium and Theorem

Definition 1 *Nash Equilibrium is a solution concept (a situation in a game) of a game that involves two or more people, in which no player has anything to gain by changing only his or her own strategy unilaterally.*

[2]http://www.cs.huji.ac.il/ noam/econcs/course/lec5.pdf

So, when everyone else is fixed to their strategy , you cannot gain anything by changing your strategy alone.
Now, let us formalize the definition. Let there be n players($P_1, P_2, ...P_n$) and player i has a strategy set S_i. If P_i chooses $s_i \in S_i$ as its strategy, then $(s_1, s_2, s_3..........s_n) \in S$ is the strategy vector.
We define S as the product of all $S_i.So, S = \otimes S_i$
The utility or payoff or benefit or cost of Player P_i is given by $u_i(s)$ where $s \in S$.
We use another notation s_{-i} that stands for the (n-1) dimensional vector of strategies player by the other (n-1) players.
A strategy vector $s \in S$ is a 'Dominant Strategy' if
$u_i(s_i, s_i') \geq u_i(s_i', s'_{-i})$ for any alternate strategy $s' \in S$.

Definition 2 *A strategy vector* $s \in S$ *is Nash Equilibrium if for all players* P_i *and each alternative strategy* $s_i' \neq s_i$*, we have*
$u_i(s_i, s'_{-i}) \geq u_i(s_i', s'_{-i})$

A game has a dominanat strategy solution if each player have a unique best strategy independent of the others.

4.1 Pure and Mixed Nash Equilibrium

A strategy defines a set of moves or actions a player will follow in a given game. A strategy must be complete, defining an action in every contingency, including those that may not be attainable in equilibrium.
A player would only use a mixed strategy when she is indifferent between several pure strategies, and when keeping the opponent guessing is desirable - that is, when the opponent can benefit from knowing the next move. [3]
To define such randomization strategies formally, let us assume that each player can pick a probability distribution over his set of possible strategies. And this choice is his 'mixed strategy'. The players are assumed to select their strategies using the probability distribution. This leads to a probability distribution of strategy vector s. Nash(1951) proved that under this extension, every game with a finite number of players , each having a finite set of strategies, has a Nash Equilibrium. This is known as Nash theorem.

Theorem 1 *Nash's theorem: Any game with a finite set of players and a finite set of strategies has a Mixed Strategy Equilibrium.*

Note that this theorem only proves the existance of an equilibrium. Finding any Nash Equilibrium is PPAD-complete. But the task of the Nash Equilibrium that maximizes the social welfare is NP-Complete [4].

[3] http://www.gametheory.net/dictionary/
[4] http://www.cs.huji.ac.il/ noam/econcs/course/lec7.pdf

5 Example Games

5.1 Prisoner's Dillemma

Two prisoners are on trial and each one faces a choice of confessing to the crime or remaining silent. If they both remain silent, the authorities will not be able to prove the major charges against them and they both will serve a short prison term of 2 years or minor offenses.If one of them confess, his term will be reduced to 1 year (for co-operating) and his confession will be used against the other prisoner, who in turn will get a 5 years sentence. But if they both confess,they both will get a prison sentence of 4 years for cooperating with the authority.
The four possible outcomes are:

P1 P2	Confess	Silent
Confess	4 4	5 1
Silent	1 5	2 2

Each of the two Prisoners has two possible strategies: to Confess or to remain Silent.The two choices of P1 correspond to the two rows and the two choices of P2 correspond to the two columns.The entries of the matrix are the costs incurred by the players in each situation. The entries of the matrix are the costs incurred by the players in each situation (left entry for the row player and right entry for the column player). This matrix is called the cost matrix.
The only stable situation in this game is that both prisoners confess. In each of the other cases , one of the players can switch from Silent to Confess and increase his payoff. But a much better outcome is if both the players stay silent. However, this is not a stable situation because in this situation, each of the players would be tempted to defect and thereby serve less time.
We say that this game has a Dominant Strategy Solution because each player has a unique best strategy independent of the other player's strategy.

5.2 Battle of Sexes

This is an example of co-ordination game. A simple co-ordination game involves two players choosing between two options, wanting to choose the same.
Consider two players, a boy and a girl, are deciding on how to spend their evening. They consider two possibilities: going to a baseball game or a softball game.The boy prefers baseball but the girl prefers softball, but they would prefer to spend the evening together rather than separately. We make a cost matrix:

Boy Girl	Baseball	Softball
Baseball	6 5	1 1
Softball	2 2	5 6

The two solutions where the two players choose differrent games is not stable since either of the two players can improve their payoff by switching their actions. On the other hand, the states where both choose the same game are stable solutions; the girl prefers the softball one and the boy prefers the baseball one.

5.3 Matching Penny Games

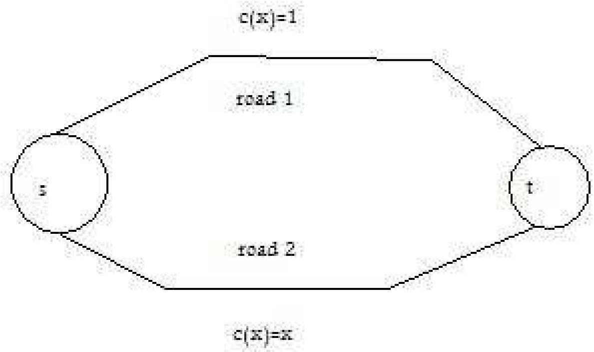
Not all games have a stable solution. The previous two exmples had stable solution in the sense that no player could gain by unilaterally changing his actions. In this example we don't have any such stability.
Two players , each having a penny, are asked to choose from two strategies- Heads(H) and Tails (T). The row player wins if both the pennies match, while the column player wins if they do not match. We show the situation in the following matrix, where 1 indicates win and -1 indicates loss.

P1 P2	H	T
H	1 -1	-1 1
T	-1 1	1 -1

The interesting thing is that at any state a loosing player can change his strategy (switching from Head(H) to Tail (T) or vice versa). So, there is no stable solution to this game. In such games players often randomly switch their strategies. But again, suppose P1 knows that P2 randomly selects between H and T. If now P1 always chooses H, he has a better Expected payoff since he knows that he will win in atleast $\frac{1}{2}$ of the cases.If on the other hand P1 knows that P2 always chooses H (or T) then his best bet is to randomly pick H(or T). But suppose, P1 has no idea about whether or not P2 is adopting a Random strategy or not. In that case, P1's best choice would be to randomize his own strategies.

5.4 Pigou's Example

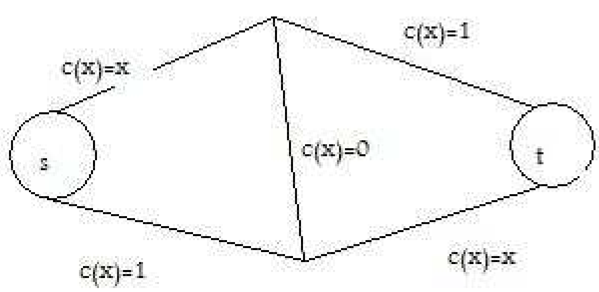
There are two roads connecting s and t. Say, the total amount of traffic is 1 unit. If x is the fraction of traffic using a road, then the delay of travelling along Road 1 is $c_1(x)$=1 . The delay of travelling along road 2 is $c_2(x) = x$.
As a start let us assume that half of the traffic takes road 1 and the other half goes through road 2.
Then the Average delay= $\frac{1}{2} \times c_1(\frac{1}{2}) + \frac{1}{2} \times c_2(\frac{1}{2}) = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$
At this point the delay along road is $\frac{1}{4}$th. So, people will prefer to take road 2.



Ultimately all traffic will take road 2 and the delay will be 1.
The Price of Anarchy (POA)= $\frac{1}{\frac{3}{4}} = \frac{4}{3}$
It turns out that the optimal delay for such networks with arbitrary functions is $\frac{3}{4}$ in the worst case. In other words, the POA and POS are the same. So, they have a unique Nash Equilibrium.

5.5 Braess's Paradox

Let us modify the previous road network so that both roads 1 and 2 now has 2 segments.



Now if we assume the same distribution of traffic ($\frac{1}{2}$ on each), the average delay is $= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$.
Now authorities add segment 1 of road 1 with segment 2 of road 2. And suppose

the delay along that link road is $c_3(x) = 0$. Interestingly, now all traffic takes the new road and the delay is $= 1 \times 1 + 1 \times 1 = 2$
By adding a new road, you expect the situation to improve (delay to decrease), but here, the delay has increased from $\frac{3}{2}$ to 2.
Therefore, POA= $\frac{2}{\frac{3}{2}} = \frac{4}{3}$.