Topics in Game Theory
Spring 2009
Lecture 2: Price of Anarchy Results for Selfish Routing in Non-atomic Games, cont.

Lecturer: MohammadTaghi HajiAghayi
Scribe: Michael Wunder

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1 Overview

Review: Congestion Games, Pigou's Example

If every player finds its best path, how does the outcome compare to the best for all players? We want to know price of anarchy.

2 Non-atomic Selfish Routing Games

- $G = (V, E)$: a directed network
- P: set of pairs (s_1, t_1), ..., (s_k, t_k)
- d_k: amount of demand in pair from s_k to t_k
- $\mathcal{P}_i$: the set of all (simple) paths from s_i to t_i
- $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i \in \mathcal{P}, f_p$
- $flow$: function $f : \mathcal{P} \to \mathfrak{R}^+$
- C_e: $\mathfrak{R}^+ \to \mathfrak{R}^+$ cost function from flow to delay which is non-negative, continuous and non-decreasing (more flow should not lead to less delay)
- f_p: flow on path p
- $f_e = \sum_{p:e\in p} f_p$: flow on edge e
- $C_p(f) = \sum_{e\in p} C_e(f_e)$ where $f_e = \sum_{p\in\mathcal{P}:e\in P} f_p$

Definition 1 *A flow f is* feasible *if for all* $i, \sum_{p \in \mathcal{P}} f_p = d_i$.

Definition 2 *A flow f is* non-atomic *if if the flow can be broken into arbitrarily small parts, whereas atomic flow means that the units of flow must be of discrete size.*

Definition 3 Non-atomic Nash equilibrium flow*: let* f *be a feasible flow for the non-atomic instance* (G, d, c)*. Then flow* f *is an equilibrium flow, if for every commodity* $i \in 1, ...k$ *and every path* p*,* $\tilde{p} \in \mathcal{P}$ *of the* $s_i - t_i$ *pair with* $f_p > 0$*, then* $C_p(f) \leq C_{\tilde{p}}(f)$*.*

The proof of existence of Nash equilibrium is derived from this definition. If another available path has less delay than the current one, it will be used instead. It is important that the flow is non-atomic because it makes the Nash equilibrium proof possible.

The objective function for flow is to minimize the total cost incurred by the traffic on G.

For feasible flow f, $C(f) = \sum_{p \in \mathcal{P}} C_p(f_p) f_p = \sum_{e \in E} C_e(f_e) f_e)$. This statement means that all units of flow incur the same total cost of flow whether we count over all paths p, or all edges e.

Theorem 1 (Existence and Uniqueness of equilibrium flows) *:*
Let (G, V, c) *be a non-atomic instance if*
a) we have at least one equilibrium flow
b) If f *and* $\tilde{f}$ *are equilibrium flow, then* $C_e(f_e) = C_e(\tilde{f_e})\ \forall e \in E(G)$
This means that POA and POS are the same, because all Nash equilibria are the same.

Proposition 1 (Characterization of optimal flows) *:*
Let (G, V, c) *be a non-atomic instance such that for every edge* e*, the function* C_e *is convex and continuously differentiable. Let* ${C_e^*}'$ *(derivative of* C_e^**) denote the marginal cost function of the edge* e*.*

Then f^* *is an* optimal *flow if and only if for every commodity* $i \in 1, ..., k$*, and every path* $p, \tilde{p} \in \mathcal{P}$ *of* $s_i - t_i$ *pair with* $f_p^* > 0$*, then* ${C_p^*}'(f^*) \leq {C_{\tilde{p}}^*}'(f^*)$*. That is, the optimal flow* f^* *is the same as the one that minimizes* $\sum_e C_e^*(f_e)$*.*

Proof: Intuitive but comes from convex optimization. The properties of a convex function on a convex set determine that the local and global minima are the same.
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2.1 Equilibrium when the marginal costs are equal

delay of path $p \in \mathcal{P}$ $C_p(f) = \sum_{e \in p} C_e(f_e)$, where $f_e = \sum_{p:e\in p} f_p$.

If you change the flow of one path to any other path, you will not save any cost across all paths. Because the rates of saving per unit flow are the same across paths where the flow is positive, they will be equal when the flow is a Nash equilibrium.

2.2 Potential Function

$C_e^*(x) = \int_0^x C_e(y)dy$

$\phi(f) = \sum_{e\in E} C_e^*(f_e) = \sum_{e \in E} \int_0^{f_e} C_e(x)dx$

This function is known a potential function. It includes the incentive of all players to change their strategy in this single function. If a change for any player improves the score of the potential, it improves the total score. That means that there is some minimum value for the potential function that cannot be changed as a result of a player improving the score, otherwise a player would choose a new strategy.

The optimal value for a game implied by the potential value method does not necessarily yield an equilibrium.

3 Price of Anarchy is 4/3 for Non-atomic Selfish Routing Games

Lemma 1 *: If f^{EQ} is an equilibrium flow in G and feasible f is arbitrary then in order for f to be the optimal flow:*

$$< C(f^{EQ}), f^{EQ} - f > \leq 0 \tag{1}$$

At equilibrium, users travel on paths minimizing $c(f^{EQ})$. If it were not the case, players could take a reverse path back from t_i to s_i, which would not conserve flow.

We are assuming affine cost $C_e = a_e x + b_e$, with $a_e, b_e \geq 0$. This setting is where Braess's paradox was initially discovered, and other important findings as well.

Proof:
Let f be the optimal flow on G.

$$\begin{aligned} C(f^{EQ}) &= \sum_e C_e(f_e^{EQ})f_e^{EQ} \le \sum_e C_e(f_e^{EQ})f_e \\ &= \sum_e C_e(f_e)f_e + \sum_e (C_e(f_e^{EQ}) - C_e(f_e))f_e \end{aligned} \tag{2}$$

Consider $\mathsf{cost}(\mathsf{OPT})$ to be cost of the optimal flow on G, and so $C_e(f_e)f_e \to \mathsf{cost}(\mathsf{OPT})$. The first term in the previous equation is $\mathsf{cost}(\mathsf{OPT})$. The second term is equal to the area of the shaded region below:

From the figure we notice that at most half of the upper triangle is covered by the shaded region. Since the area of the whole rectangle is equal to $\sum_e C_e(f_e^{EQ})f_e^{EQ}$, we conclude the following:

$$\sum_e (C_e(f_e^{EQ}) - C_e(f_e))f_e \le 1/4 C_e(f_e^{EQ})f_e^{EQ} \tag{3}$$

From these equations (2) and (3), we have:

$$\begin{aligned} \frac{3}{4}\mathsf{cost}(f^{EQ}) &\le \mathsf{cost}(\mathsf{OPT}) \\ \mathsf{cost}(f^{EQ}) &\le \frac{4}{3}\mathsf{cost}(\mathsf{OPT}) \\ \frac{\mathsf{cost}(f^{EQ})}{\mathsf{cost}(\mathsf{OPT})} &\le \frac{4}{3} \\ \text{Price of Anarchy} &= \frac{4}{3} \end{aligned}$$

We are only worried about $f_e < f_e^{EQ}$ since C_e is non-decreasing.

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For more detail about these theorems and proofs, see Chapter 18 from [2] or the original paper [3]. A version of the 4/3 proof found above was found in [1] and adapted and simplified by the lecturer.

4 Final Projects

Look at topics on website and choose one that appeals to you as a final project. Please come prepared to discuss your chosen topic. Group size may be two and perhaps three. Most topics can be covered by a single person.

References

[1] J. Correa, A. Schulz, and N. Stier-Moses. Selfish routing in capacitated networks. *Mathematics of Operations Research*, 29(4):961–976, 2004.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani. *Algorithmic Game Theory*. The MIT Press, 2007.

[3] T. Roughgarden and . Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.