Topics in Game Theory
Spring 2009
Lecture 3: Market Equilibrium

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1 Overview

Until now we have studied Nash equilibrium. We look at Market Equilibrium or Market Clearance Prices. We study two market equilibrium models, the Fisher setting with linear utilities and the Arrow-Debreu model with linear utilities. Then we see an application of the Fisher model to wireless networks.

A market consists of a set of sellers who have goods to sell and buyers who would like to buy some of the goods. A utility function represents the importance of an item to an agent. The prices at market equilibrium ensure that the buyer finds the best set of items to buy and that all goods are sold. Thus, equilibrium satisfies all agents.

2 Fisher Model

- B is the set of buyers, each with budget B_i. $|B| = m$.
- G is the set of goods, each with quantity q_j. $|G| = n$.
- $u_{ij} > 0$ is the utility of buyer i for one unit of good j.
- x_{ij} is the amount of good j bought by buyer i.
- Each buyer has a linear utility function $u_i = \sum_{j=1}^{n} u_{ij} x_{ij}$.

We consider a market consisting of buyers and divisible goods. The money possessed by each buyer, the linear utility function of each buyer and the amount of each good are specified. The problem is to compute prices for the goods that optimally satisfy each buyer and for which there is no deficiency or surplus of any good. That is, the market clears.

Definition 1 *A* MARKET EQUILIBRIUM *or* MARKET CLEARANCE PRICES *is a vector* p_i *such that*
a) it maximizes the utility u_i *of buyer* i *within his budget, i.e.,* $\forall i, \sum_j p_j x_{ij} \le B_i$
b) the demand equals the supply of each good, i.e., $\forall j, \sum_i x_{ij} = q_j$

Theorem 1 *Under the mild assumption that each good has a potential buyer,*
a) market clearance prices do exist
b) the set of equilibrium allocations is convex
c) equilibrium uitlities and prices are unique

We make the assumpation that for each good j, there exists a buyer i such that $u_{ij} > 0$. This condition ensures that the equilibrium price of each good is positive.

Proof:

The Fisher setting can be formulated as a convex program; using the Eisenberg-Gale approach we solve for the unique set of prices that form an equilibrium.

We can assume w.l.o.g. that $\forall j, q_j = 1$ (we have a unit supply of each good), and that $\forall i, \forall j, u_{ij}, B_j$ are rational integers. That is, we can scale the values.

The problem can be represented as the following Eisenberg-Gale convex program:

maximize

$$\prod_{i \in B} u_i^{B_i}$$

which is equivalent to
maximize

$$\sum_{i \in B} B_i \log u_i$$

subject to

$$u_i = \sum_{j \in G} u_{ij} x_{ij}, \forall i \in B$$

$$\sum_{i \in B} x_{ij} \le 1, \forall j \in G$$

$$x_{ij} \ge 0, \forall i \in B, \forall j \in G$$

In a convex programming problem, one seeks to minimize a convex function over a convex body. Not all convex functions are linear. So convex programming is more general than linear programming. Convex programming can be solved by polynomial algorithms. Since the aforementioned objective function is concave, we can negate it to form a convex function that we would like to minimize. Thus, we can find the optimal solution in polynomial time. Similar to the dual of a linear program, convex programms have Karush, Kahn, Tucker (KKT) conditions.

The convex program has these KKT conditions:

(i)

$$j \in G, p_j \geq 0$$

(ii)

$$\forall j \in G, p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$$

(iii)

$$\forall i \in B, \forall j \in G, \frac{u_{ij}}{p_j} \leq \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$$

(iv)

$$\forall i \in B, \forall j \in G, x_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$$

For each good j, there is a buyer i with $u_{ij} > 0$. From the four KKT conditions we can derive the existence of market clearance prices.
by (iii),

$$p_j \geq \frac{B_i u_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} > 0$$

by (iv),

$$\frac{u_{ij}}{p_j} x_{ij} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i} x_{ij}$$

Thus,

$$\frac{B_i u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = p_j x_{ij}$$

Sum over j,

$$\frac{B_i \sum_{j \in G} u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = \sum_{j \in G} p_j x_{ij}$$

which reduces to

$$B_i = \sum_{j \in G} p_j x_{ij}$$

We can compute market clearance prices, so we see that (a) they actually exist. We can use properties of convex programming to prove that (b) the set of equilibrium allocations is convex and (c) the uniqueness of equilibrium utilities and prices. ■

3 Arrow-Debreu Model

Also known as the WALRASION model and as the EXCHANGE model, the ARROW-DEBREU model is a generalization of the FISHER model. Each agent wants to sell his goods to purchase other goods. In the Fisher model, each agent is designated to act as either a buyer or a seller. In the exchange model, the agents are not separated into groups of buyers and sellers. Rather, every agent functions as both a buyer and a seller.

- A is the set of AGENTS, each with budget B_i. $|A| = m$.
- G is the set of GOODS, each with quantity q_j. $|G| = n$.
- $u_{ij} > 0$ is the UTILITY of agent i for one unit of good j.
- x_{ij} is the amount of good j bought by agent i.
- Each agent has a linear UTILITY FUNCTION $u_i = \sum_{j=1}^{n} u_{ij} x_{ij}$.
- Each agent i begins with an initial endowment of goods, $e_i = (e_{i1}, e_{i2}, ..., e_{in})$.

  We may assume w.l.o.g that the total amount of each good is unit, i.e., $1 \leq j \leq n, \sum_{i=1}^{m} e_{ij} = 1$.

We consider a market consisting of agents and divisible goods. The amount of each good possessed by each agent and the linear utility function of each agent are specified. The goal is to compute a price vector $p = (p_1, p_2, ..., p_n)$ for the goods that maximizes the utility of each agent and for which all goods are traded.

Theorem 2 *In the Arrow-Debreau model, there exists a set of market clearance prices and the corresponding assignment that maximizes the utilities and indeed it is unique.*

This theorem can be proven in three different ways:

1. convex programming based (Eisenberg, Gale 1957)
2. primal-dual based (Garg, Kapoor 2003)
3. auction based (Garg, Kapoor 2003)

A Fisher market with linear utilities, n goods, and m buyers reduces to an Arrow-Debreu market with linear utilities, $n+1$ goods, and $m+1$ agents as follows.

- The set of $n+1$ goods consists of n goods and money.
- The set of $m+1$ agents consists of m buyers and the seller.
- Each buyer's initial endowments are money, nothing else. These m agents have zero utility for money and positive utility for everything else.
- The seller's initial endowment is the set of goods, and no money. This agent has positive utility for money and zero utility for all other goods.

4 Application to Distributed Load Balancing

4.1 What is the problem?

Wireless devices communicate with an Access Point (AP) to gain access to a network. Devices connect to the AP that offers the strongest signal. Access Points have limited capacity and variable transmission power. Because of these constraints, local concentration of devices creates congestion at several APs. We would like a techniuqe to efficiently assign clients to APs that does not require cooperation from both the APs and clients.

The cell breathing heuristic uses concepts from game theory. In the proposed algorithm, clients continue to associate with an AP that offers the strongest signal. An AP manage its load by adjusting its advertised transmission power. In this way, the APs coverage area expands or contracts, adapting to client demands. Thus, traffic is balanced across the network. The appropriate power level should be assigned at APs to automatically achieve load balancing.

4.2 How does it relate to market equilibrium?

There is a correspondence between market equilibrium in a Fisher setting with linear inequalities and distributed load balancing.

Market Equilibrium	**Load Balancing**
Seller	AP
Buyer	wireless client
Goods	network connectivity
Supply	capacity of AP
Price	power at AP
Utility	received signal strength function
market clearance	either all clients served or all APs saturated

We can get inspiration from various algorithms for the Fisher setting to develop solutions for wireless networks. Both areas seek to arrive at a stable system that maximizes the satisfaction of its users.

4.3 What are the differences between the problems?

	Market Equilibrium	**Load Balancing**
Demand	price dependent	power independent
Is demand splittable?	Yes	No
Complexity	P	In continuous case: P In discrete case: APX-Hard
Equilibrium	clears both sides	clearance on either AP side or client side
	can be computed	may not exist

The algorithms in game theory that provide intuition to the network application can be solved precisely. In the continuous case, when the APs can adjust their power to any level, this is maintained. Yet, in the discrete case, the assignment of clients to Access Points may never reach an equilibrium. Nonetheless, we can achieve approximation algorithms that are almost optimal.

The theory of market equilibrium provides us with meaningful mechanisms to solve a seemingly unrelated problem, syncronized distributed load balancing in wireless networks. This novel approach provides straightforward algorithmic solutions.

5 References

For further details about these concepts and methods, see chapter 5 of *Algorithmic Game Theory*. The paper entitled "The Gale and EisenbergMarket Equilibrium via a Primal-Dual Algorithm for a Convex Program", by Nikhil R. Devanur, Christos H. Papadimitriou, Amin Saberi, and Vijay V. Vazirani, presents a polynomial algorithm for computing market equilibrium by extending the primal-dual schema. Prof. Mohammad Taghi Hajiaghayi's paper entitle "Cell Breathing in Wireless LANs: Algorithms and Evaluation" (MobiCom 2007) describes the application of market equilibrium to wireless networks.

6 Projects

Each group should email the instructors to confirm their project topic. Mention the course name "CS 514" in the subject.