Topics in Game Theory
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Lecture 7: Complexity of finding Nash Equilibria

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1 Overview

Nash's theorem guarantees existence of mixed Nash equilibrium, but when do pure equilibria exist and how hard are they to find? Finding pure Nash equilibrium is easy for 2 players, a simple brute force algorithm checking all the entries of the payoff table will do it. However in general for n players with the payoff table given in some implicit way, the number of possible strategy combinations is exponential. Furthermore if we want an efficient algorithm we can't go to full generality of the payoff functions either. (Example: player 1 wins if TM halts on input) So what are the good games to think about? [1]

2 Congestion games

There is a famous and well-studied class of games (and, in fact, one with obvious affinity to networks) that is guaranteed to have pure Nash equilibria: the *congestion games*.

2.1 Complexity classes

The complexity class FNP is the function problem extension of the decision class NP. An instance of this class is a relation between the problem instances and their solutions.

Definition 1 (FNP)

$$\mathsf{FNP} = \{\mathsf{R} \subseteq \Sigma^* \times \Sigma^* | \textit{ given (s,t) can be checked in poly-time if } \mathsf{R}(\mathsf{s},\mathsf{t}) \textit{ holds}\}$$

An example is the extension of the SAT problem FSAT, where s is the formula and t is a satisfying assignment. A subset of this is the class TFNP where a solution is guaranteed to exist:

Definition 2 (TFNP)

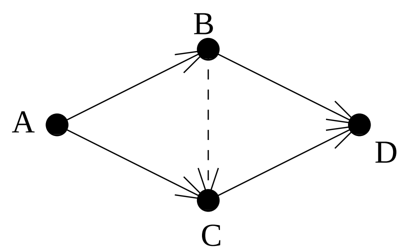
$$\mathsf{TFNP} = \{R \in \mathsf{FNP} | \forall s. \exists t. R(s,t)\}$$

Note that unlike NP, now we need 2 mappings for a reduction $R \preceq R'$, one for mapping an input x of R to an input x' of R' and another one for mapping the output y' of R' to an output of R.

Now suppose that the inclusion holds in the other direction also, so $\mathsf{FNP} \subseteq \mathsf{TFNP}$ ($\mathsf{TFNP} = \mathsf{FNP}$). But then $\mathsf{FSAT} \in \mathsf{FNP} \subseteq \mathsf{TFNP}$ would imply $\mathsf{SAT} \in \mathsf{coNP}$ which would eventually mean $\mathsf{NP} = \mathsf{coNP}$.

2.2 Networks

Consider the congestion game given by the following graph (Braess's paradox):



We have 100 people who would like to go from A to D using only the edges drawn with solid line and we have the following delay/cost function for the edges: $c_{AB}(x) = x$, $c_{AC}(x) = 100$, $c_{BD}(x) = 100$ and $c_{CD}(x) = x$. The equilibrium in this case is if 50 people use the ABD path and the other 50 the ACD path resulting in a delay of $50 + 100 = 150$ for everyone.

However by adding a new edge, BC with 0 delay, the equilibrium will move to a state with higher delay, as everyone will use the $ABCD$ path, with delay $100 + 0 + 100 = 200$.

Theorem 1 *Every congestion game has a pure Nash equilibrium*

Proof: Let's define a potential function $\Psi = \sum_{e \in E} \Psi_e$ and $\Psi_e = \sum_{i=1}^{x_e} c_e(i)$ where x_e is the traffic on edge e. Note that altough the potential is always less or equal than the total delay it has the property that if someone changes edges the change in the potential is the same as in the delay of the deviating player. So the local minimum point of the potential is a state where no one can change and gain, which is exactly the definition of pure Nash equilibrium. ∎

2.3 PLS

The complexity class TFNP is a semantic class, meaning that it seems there is no complete problem for it.

However if we have a tool like a potential function we can use the PLS class [2] which does have complete problems. PLS is a subclass of TFNP and the letters stand for polynomial local search. It contains problems finding a local minimum given the search space and a polynomial time cost function.

2.3.1 A PLS-complete problem

The **POS-NAE-3SAT** problem is a variation of the **3SAT** problem. We have conjunction of clauses, but now a clause means that at least one of its at most 3 variables has different value than the others. The cost of a solution is the sum of the weights of the unsatisfied clauses and we'd light to get a solution by which we can not get a better one by flipping one of the variables.

Example 1 $\neq (x, y, z) \land \neq (x, v, w) \land \neq (v, u, x)$ *and the corresponding weights are 5, 2 and 4.*

Theorem 2 *The congestion games are also complete for the* PLS *class.*

2.3.2 PLS-completeness of congestion games

Proof: We are showing the completeness by giving a reduction **POS-NAE-3SAT**$\preceq_{\mathsf{PLS}}$**CG**.

For each clause c with weigth w add 2 resources e_c, e'_c and define their delay in the following way:

$$d(e_c) = d(e'_c) = \begin{cases} 0 & \leq 2 \text{ players using it} \\ w & \text{otherwise} \end{cases}$$

For each variable x add a player with the following 2 strategies:

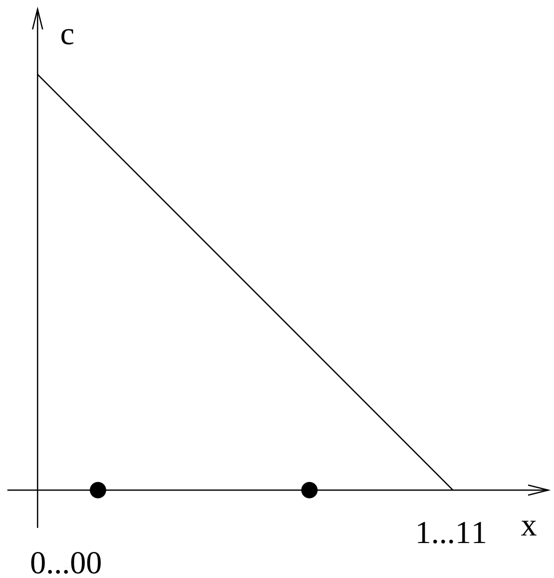
$$\{e_c | \forall c \ x \text{ appears in}\}$$

$$\{e'_c | \forall c \ x \text{ appears in}\}$$

Now any Nash equilibrium of the congestion game is a local optimum of the **POS-NAE-3SAT** problem. ■

Note that we are only looking for a local optimum and not for one closest to a given pont. To illustrate the difference consider the following problem: Suppose we have a **SAT** formula $\Phi(\overrightarrow{x})$ on n bits and a cost function:

$$c(\overrightarrow{x}) = \begin{cases} 2^n & \text{if } \Phi(\overrightarrow{x}) = 0 \\ 0 & \text{otherwise} \end{cases}$$



Now we can easily give a local optimum in constant time: 1...11 while computing the local optimum closest to 0...00 would solve the **SAT** problem.

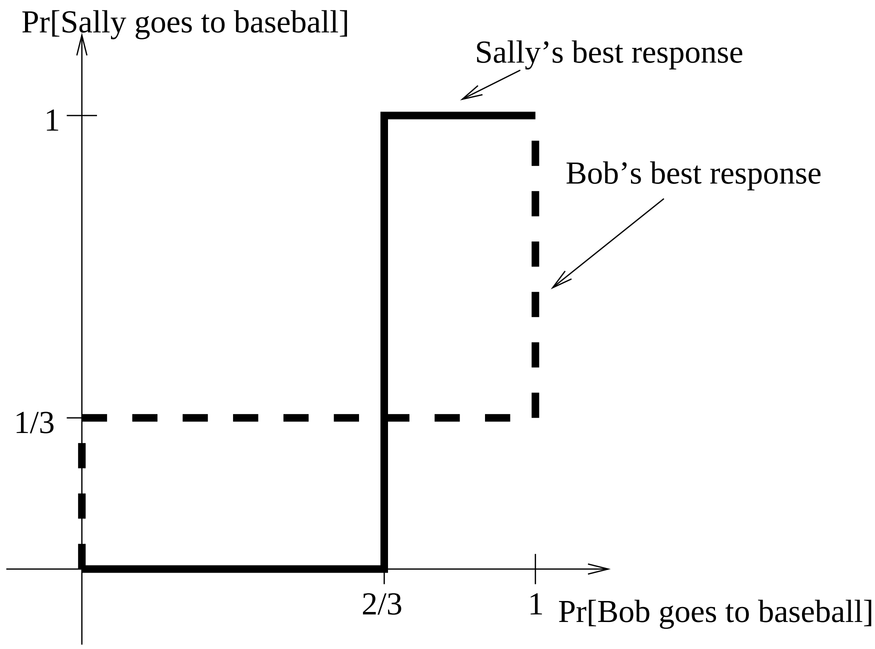
3 Mixed Nash equilibrium

3.1 Battle of sexes

For a quick recap let's see the Battle of sexes example: Sally and Bob want to go out. Sally prefers going to the softball game while Bob would rather go to the baseball game but both of them wants to be with the other one. The payoff table looks like the following:

Bob\Sally	Baseball	Softball
Baseball	2,1	0,0
Softball	0,0	1,2

It's easy to see that this game has 2 pure Nash equilibria: if both of them go to the same place either to softball or baseball. Now in case of mixed strategies Sally's best response is to go to the softball game if Bob goes to baseball with probability no more than $\frac{2}{3}$ and go to baseball otherwise. Similarly Bob's best response is to go to softball if Sally goes to baseball with probability no more than $\frac{1}{3}$ and go to softball baseball otherwise.



There best response curves have 3 intersections: 2 at the pure Nash equilibria and the third corresponds to a mixed Nash equilibrium. In general each intersection of the best response curves is a Nash equilibrium.

3.2 Brouwer's fixed point theorem

Nash's theorem that each game has a mixed Nash equilibrium can be shown using Brouwer's fixed point theorem.

Theorem 3 (Brouwer's fixed point theorem) $\forall S \subset \mathbb{R}^n$ *convex and compact set and continuous function* $f : S \to S$ $\exists x^*$ *"fixed point" such that* $f(x^*) = x^*$.

Take the following function:

$$f(\vec{x}, \vec{y}) = (\vec{x}', \vec{y}') = \begin{cases} \vec{x}' \text{ is player 1's best response to } \vec{y} \\ \vec{y}' \text{ is player 2's best response to } \vec{x} \end{cases}$$

Clearly the fixed point of this function is a mixed Nash equilibrium.

$$g(\vec{x}, \vec{y}) = \begin{cases} x_i' \leftarrow \frac{x_i + c_i(\vec{x}, \vec{y})}{1 + \sum_{K \in S_1} c_k(\vec{x}, \vec{y}))} \\ y_i' \leftarrow \frac{y_i + d_i(\vec{x}, \vec{y})}{1 + \sum_{K \in S_2} d_k(\vec{x}, \vec{y}))} \end{cases}$$

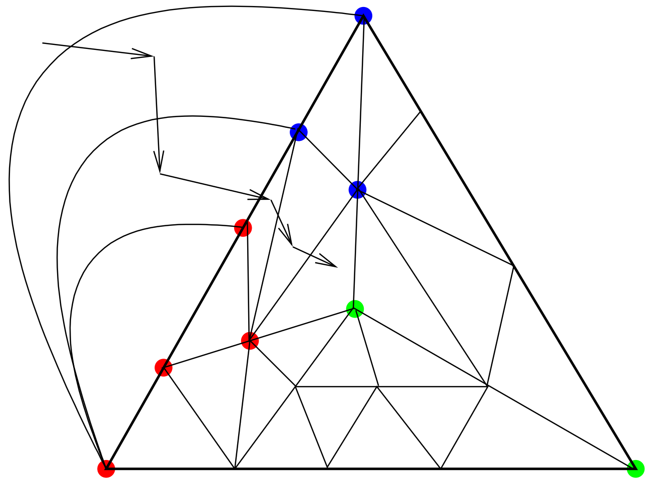
where S_i is the set of ith player's strategies.

$$c(\vec{x}, \vec{y}) = \begin{cases} 0 \text{ if playing just } i \text{(with probability 1) is worse response to } \vec{y} \text{ than } \vec{x} \\ \text{otherwise the improvement from playing just } i \end{cases}$$

$\mathrm{d}(\vec{x}, \vec{y})$ is defined similarly.

The combinatorial analog of Brouwer's fixed point theorem is Sperner's lemma:

Lemma 1 (Sperner's lemma) *A triangle and its triangulation is given. Each of the vertices of the big triangle has a unique color and each vertex on an edge of the big triangle can only have the color of one of the two colors of the endpoints of its edge. In any such scenario there exists a trichromatic triangle.*



Proof: Add "missing edges" as seen on the figure and think of these new areas as other triangles (easy to see that none of these will be trichromatic). Start from the outer area and always only go across edges that are colored say blue-red. The outer area has only one edge colored blue-red therefore it has only one outgoing edge. All the triangles have in degree and out degree both at most 1 and as the outer area only had an outgoing edge, there must be an other triangle with only an incoming edge which means a trichromatic triangle. ■

3.3 PPAD

The complexity class PPAD (Polynomial Parity Arguments on Directed graphs) introduced in [3] is a subclass of TFNP where the solution is guaranteed by parity argument: we define a chain with predecessor and successor functions.

The previous problem and finding a Nash equilibrium in a normal form game of two or more players with specified utilities, is in PPAD, and proved to be complete for PPAD with 2, 3 and 4 players in [4].

In [5] an oracle separation is also shown proving that PPAD is not a superclass of PPP.

4 Homework

When we enter a query in google some ads will appear on the right hand side. There are 3 participants in this setting: users, search engine and advertisers. Whether an ad is shown or not is the choice of the advertiser.

Example 2 *Gerlanda Pizza wants its ad to be displayed when the search comes from New Jersey, in the evening hours and the query contains party, or movie, but probably not for pizza or gerlanda.*

The task is to pick a company, and design a campaign for it.

References

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