Topics in Game Theory
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Lecture 8: Auction Theory

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1 Overview

In this lecture we study auctions. In the basic setting there are n bidders, bidding for a single item. While the single seller sells the item to one of the bidders at a price such that the mechanism maximizes 'social welfare'. More Generally, there are k identical items to sell, while each bidder is interested in only one item. The seller chooses a set of k bidders, again to maximize the social welfare. We discuss Vickery's second price auction for the single item case and the Vickrey-Clarke-Groves (VCG) mechanism for k items auction. We argue that these mechanisms maximize social wefare and are truthful.

2 Single item auction formulation

Single item auction is defined as follows:
A single seller who wants to sell an item.
A set of alternatives, $A = \{1, 2, \ldots, n\}$
A set of players, I, $|I| = n$.
Set of Valuation functions $v_i : A \to \mathbb{R}$ for each i.
Set of bids b_i not necessarily equal to v_i.

The seller choses one of the alternatives, i.e. select one of the bidders, that the item will be sold to (i-wins).
The valuations is a value that bidder i assign to the item if alternative j is chosen, this is private information of the bidder.
We define utility (also known as profit or benefit) of bidder i, $u_i = v_i(a) - p_i$, where p_i is the amount that auctioneer (seller) charges bidder i. Throughout we assume that bidder has quasi-linear utility as shown above. We also assume

that the bidder has 0 value if the item is assigned to any other bidder and that there is no collusion among bidders.

$$u_i = \begin{cases} v_i - p_i & \text{if item is allocated to bidder } i, \\ 0 & \text{otherwise} \end{cases}$$

We say that a mechanism for auction is truthful if bidders have no incentive to not reveal his true value for the item.

Each bidder places a bid b_i for the item, and the goal of the auctioneer is to design a mechanism that will find an allocation that maximize the social welfare, where the social welfare (also known as efficiency) is defined as the sum of utilities of all bidders, i.e. efficiency $= \sum_i u_i$.

3 Vickery 2nd Price Auction

One natural choice to achieve social welfare is to give the item to the highest bidder for free. This clearly maximizes the social welfare but it is obviously not truthful, as every bidder will bid much higher than his true value.

Alternatively, if we sell the item to the highest bidder at the price equal to his declared value, the utility of every player will be 0. Also, it clearly gives an incentive to the highest bidder i to bid $b_i < v_i$ that will still be the highest bid. In this case his utility will be greater than that if he bids v_i. So i has a choice to bid as low as his bid is still higher than the second highest bidder. Hence it is not a truthful mechansim.

In what follows we give the Vickery's Second Price Auction and prove that it is truthful.

Definition 1 (Vickery's 2nd Price Auction) *Let the winner be bidder i whose bid is the highest and the price will be the second highest bid, $p^* = \max_{j \neq i} b_j$.*

Theorem 1 (Vickery's 2nd Price Auction is truthful) *For every $v_1, v_2, \ldots, v_n$ and every $v'_i \neq v_i$, Let u_i be the utility of bidder i if he bids v_i and let u'_i be the his utility if he bids v'_i. Then $u_i \geq u'_i$.*

Proof: Assume that bidder i wins by bidding v_i, and the second highest price is p^*, in this case $u_i = v_i - p^*$. Assume now that i bids v'_i and still wins. By the definition of mechanism i will win only if $v'_i > p^*$ and the price he will pay will still be p^*, So utility will remain the same, hence $u_i \geq u'_i$.

In case i bid less than p^* he will lose the item, so his utitlity will be 0, hence $u_i \geq u'_i$. ■

4 Multiple items

In this case the autioneer has to sell k identical items and each bidder is interested in one item. So the alternatives are k-subsets of $\{1, 2, \ldots, n\}$ and the rest of the setting remains the same. We present Vickrey-Clarke-Groves mechanism and prove its truthfulness and efficiency. Denote by v_{-i} the vector of all values except the value of player i.

Definition 2 (VCG Mechanism) *A mechanism* $(f, p_1, p_2, \ldots, p_n)$ *is called VCG mechanism if*
$f(v_1, v_2, \ldots, v_n) = \arg\max_a \sum_{a \in A} v_i(a)$.
and for some functions $h_1, h_2, \ldots, h_n$, $h_i : v_{-i} \to \mathbb{R}$, $p_i(v_1, v_2, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(a)$ *where* $a = f(v_1, v_2, \ldots, v_n)$.

Theorem 2 *Every VCG Mechansim is truthful.*

Proof: We prove that for every player i with value v_i, his utility when bidding v_i is at least as good as his utility when bidding any $v_i' \neq v_i$
Let u_i and u_i' be the utility of player i if he bids v_i and v_i' respectively. Suppose that $f(v_i, v_{-i}) = a$ and $f(v_i', v_{-i}) = b$ i.e. a is the alternative chosen if player i reveals his true value while b is the alternative chosen if he lies.
The utility of i, $u_i = v_i(a) - p_i(v_i, v_{-i}) = v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) = \sum_i v_i(a) - h_i(v_{-i}) \geq \sum_i v_i(b) - h_i(v_{-i}) = v_i(b) + \sum_{j \neq i} v_j(b) - h_i(v_{-i}) = v_i(b) - p_i(v_i, v_{-i})$. ■
The inequality follows from definition of VCG mechanism.
This mechanism depends on the functions h_i; the auctioneer may be paying instead of being paid. So the choice of functions h_i is critical.

Definition 3 (Clarke pivot rule) $h_i(v_{-i}) = max_{b \in A} \sum_{i \neq j} v_i(b)$.

By definition of VCG mechanism, with this choice of functions h_i, the payment of player i, $p_i(v_1, \ldots, v_n) = max_{b \in A} \sum_{i \neq j} v_i(b) - \sum_{j \neq i} v_j(a)$ where $a = f(v_1, v_2, \ldots, v_n)$.

This choice of h_i ensures that

- Payments are always non-negative, (the auctioneer does not pay anything).
- All palyers utilities are non-negative.
- If a player does not win an item he pays 0.
- A players pays the demage he caused, (the difference between the social welfare of other players with and without player i's participation.
- This only works if all players values are non negative.

It is easy to see that in case $k = 1$ this is simply Vickery's second price auction, and in case $1 < k < n$ and that every player is interested in only one item, this mechanism allocates the items to k highest bidders at price equal to $k + 1$'st highest bid.