Automated Online Mechanism Design and Prophet Inequalities

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In this presentation

- Mechanism design
- Automated Mechanism Design (AMD)
- Automated Online Mechanism Design(OMD).
- Prophet Inequality and OMD

Cutting a pie fairly: Mechanism design issues

- Conflicting Interest of multiple agents
- Manipulability: An agent may have an incentive to misreport its preferences.
- The aim is to aggregate preferences to choose a socially desirable outcome.

Matrics of Mechanism Design

- Efficiency:The efficiency of a mechanism is the combined utility of all buyers for the allocation.
- Revenue:the revenue is the sum of the payments collected from agents.

Mechanism Design

- Mechanism design started off as a subfield of Economics that attempts implementing desired social choices in a strategic setting.
- It is the art of setting rules of the game to achieve a desired outcome.
- In an auction , the goal of mechanism design may be to maximize the revenue of the seller or to maximize the chance of selling an item to the buyer who values it most.

Game Theory vs. Mechanism design[1]

- Using game theory, mechanism design can explicitly model how prices are set.
- Mechanism design theory shows which mechanisms are optimal for different participants.
- Incentive- compatibility: There is no incentive for hiding the truth.
- Revelation Principle: any equilibrium outcome of an arbitrary mechanism can be replicated by an incentive-compatible direct revelation mechanism. This theorem greatly simplified the mechanism design problem.

[1] Mechanism Design Theory: Compiled by the Prize Committee of the Royal Swedish Academy of Sciences

Manual Mechanism Design

- The designer uses experience and intuition to hypothesize that a certain rule set is desirable in some ways, and then tries to prove that this is the case.
- Alternatively, the designer formulates the mechanism design problem mathematically and characterizes desirable mechanisms analytically in that framework
- It makes assumption about agents and the mechanism works optimaly only for that setting.

Automated Mechanism Design[2]

The mechanism is computationally created for the specific problem instance at hand.

- Can yield better mechanisms than the ones known to date by capitalizing on the particulars of the setting.
- Applies beyond the problem classes studied manually to date.e.g. revenue-maximizing combinatorial auction
- Can circumvent seminal economic impossibility results
- Shifts the burden of design from man to machine.

[2]Thomas Sandholm, Automated mechanism design: A new application area for search algorithms.

Offline vs. Online Mechanism Design

- Mechanism design has traditionally focused on settings where all the agents are present upfront.
- In an online setting, agents may arrive and depart dynamically.
- How do we make feasible decisions given the uncertainty about future? This makes the problem of online mechanism design (OMD) fundamentally different from traditional mechanism design(MD).

Some aspects of Online Mechanism Design (OMD)

- Decisions must be made without any knowledge about types of agent who have not yet arrived.
- Agents may misrepresent their arrival and departure times along with their valuations.
- Only partial misreport of agent types may be possible. i.e. it may be impossible for an agent to report an arrival time that is earlier than the true arrival time.

Contributions by (Hajiaghayi, Kleinberg, Sandholm 2007)

- Design online mechanism when the number of bidders is not known in advance but the seller has distributional knowledge about the bid values
- Algorithm for mechanism design where the number of bidders is known at least probabilistically.
- Relationship between these mechanism design problems and *Prophet Inequality*

Setting

- k units to sell, n agents(or bidders)- each bidder wants 1 unit.
- Each agent has a type (a_i, d_i, v_i) for $i, 1 \leq i \leq n$.
- p_i is the price that agent i pays.
- If a bidder gets an item his utility is $u_i = v_i - p_i$
- Agents may report a (false) type $(\hat{a}_i, \hat{d}_i, \hat{v}_i)$ satisfying $\hat{a}_i \geq a_i$.

Mechanism

- *allocation rule* $q_i(t, \vec{\theta})$ (where t is the time and $\vec{\theta}$ is the vector of reported types)
- $q_i(t, \vec{\theta})$ depends only on the types reported by agents with $\hat{a}_i \leq t$
- *payment rule* $p_i(\vec{\theta})$)

Measures of Mechanism

- *efficiency* $= \sum_i q_i v_i$
- *revenue* $= \sum_i p_i$
- The mechanism is ρ-competitive if the expected efficiency (revenue) is at least $\frac{1}{\rho}$ times the expeted value of optimum efficiency (revenue).

Seller's Information

We make two different types of assumptions about the seller's distributional information:

- **Full Information:** The seller has distributional information about valuations and market size.
- **Partial Information:** The seller has distributional information about valuations but not the market size.

Interesting Cases

The following cases are of interest:

- **Independent bids:** The bids are all independent.
- **Independent bids, fixed n:** The bids are independent but the value of n is fixed.
- **i.i.d. bids** The bids are independent and identically distributed.
- **dependent bids, unkown n** The bids are independent and identically distributed.

Prior work

- The first paper on online auctions was in 2000 (Lavi & Nisan 2000)
- Algorithmic Game Theory: Chapter 16
- Assumes :
  - the agents arrive in a pre-determined order which is not under control
  - The agent's only private information is her value.
- Some online mechanisms (e.g. (Awerbuch, Azar, & Meyerson 2003; Lavi & Nisan 2000)) are strategyproof against agents misstating their arrival or departure time because they are based on prices which do not decrease over time.

Prior work

For K identical goods, (Hajiaghayi, Kleinberg, & Parkes 2004) gives a constant competitive mechanism.

- Agents arrive and depart dynamically
- Each agent has (arrival time, departure time, value).
- The number of agents is known in advance

Papers studing the case for re-usable goods(e.g. processor time)

- (Hajiaghayi et al. 2005)
- (Porter 2004)
- (Lavi & Nisan 2005)

Prior work

- Parallel and independent work by (Pai & Vohra 2006)
- (Mahdian & Saberi 2006) assumed an unknown n for perishable goods.
  - Gives a revenue-maximizing auction algorithm
  - But they don't consider any distributional knowledge of the bid values

Unknown n, i.i.d bids

- No constant competitive mechansim in this scenario
- Not even for $k = 1$ and instantaneous bidders.

Counter-example for k=1

No C-competitive mechanism, for $C > 1$.

- Bids $\in \{0, 1, C, C^2, C^3, ...\}$ with distribution f
- A sequence of numbers $t_1, t_2, ...$, such that with high probability max of t_r independent samples from $f = C^r$.
- Find an r such that
  $\Pr(\text{Mechanism sells item at } t,\ t_{r-1} \leq t \leq t_r) < 1/C$
- Set $n = t_r$

The optimum efficiency (revenue) is C^r (by definition).
The Mechansim, with high probability, either

- does not sell the item; efficiency (revenue) $= 0$
- sells the item before t_{r-1}; efficiency (revenue) $\leq C^{r-1}$.

Counter-example for k=1

No C-competitive mechanism, for $C > 1$.
Following construction achieve the properties

- $p(C^r) = 2^{-2^r} - 2^{-2^{r+1}}$ for $r = 1, 2, 3 \ldots$
- $t_r = C \cdot 2^{2^r}$

Unknown n: $k > 1$ $\min\{O(\log h), O(\log n(\log\log n)^2)\}$-competitiveness

h is the ratio of the maximum bid to the minimum bid.
$O(\log h)$ competitive mechanism:

- Make $\log h$ classes of the bid values like this: $[1, 2), [2, 4), [4, 8), \ldots, [b_{max}/2, b_{max}]$.
- Choose a class at random, set Price p = min of chosen class
- Sell to first k bids $\geq p$, (if any)
- Probability of choosing the class containing optimum price $= O(1/\log h)$, So $2p \geq P_{OPT}$
- Hence $O(\log h)$-competivieness

Unknown n: $k > 1$ $\min\{O(\log h), O(\log n(\log\log n)^2)\}$-competitiveness

$O(\log n(\log\log n)^2)\}$-competitive mechanism:

- Guess the value of n as a random power of 2
  - $n' = 2^r$ and $Pr(r \geq s) = 1/(\log(s))$
- Sell at most k units among the first 2^r bidders using a C-competitive mechanism.
- $Pr(n/2 \leq 2^r \leq n) \geq \Omega((\log n)^{-1}(\log\log n)^{-2})$

n has Bounded support

For a finite maximum number of bidders, and a non-decreasing price sequency, (Hajiaghayi, Kleinberg, Sandholm 2007) gives a dynamic algorithm that uses backward induction on time to calculate the best expected revenue (efficiency) that can be obtained

n' is the number of bidders seen so far,

k' is the number of remaining units to sell,

and q' is the last (or the largest) price at which we sold a unit so far.

Then $D[n', k', q']$ keeps the best expected revenue (efficiency) that we could obtain at this stage.

The Algorithm

Algorithm OPTMech

for $n' = N$ down to 0 **do**
  for $k' = 1$ to k **do**
    for $q' = Q$ down to 0 **do**
      let $p_{>n'}$ be the probability from distribution N of having more than n' bidders conditioned on having seen n' of them so far
      let $p \geq q''$ be the probability from distribution $\mathcal{Q}_{n'}$ that the n' th bidder has valuation q'' or higher
      set reserved price $R[n', k', q'] = q''$ where $q' \leq q'' \leq Q$ maximizes: $(1 - p_{>n'})p_{>q''}B_{q''} + p_{>n'}((1 - p_{>q''})D[n + 1, k, q''] + p_{>q''}(B_{q''} + D[n' + 1, k - 1, q'']))$ where set $B''_q = q''$ to maximize revenue or set $B_{q''} = \sum_{x=q''}^{Q} p_x x$to maximize efficiency ($p_x$ is the probability that the n'th bidder has valuation exactly x)
    end for
  end for

Known distribution over n: Unbounded support

For 1-unit auction if n is drawn from a distribution with *non-increasing hazard rate*, (i.e. a distribution such that the function $\phi(t) = Pr(n = t | n \geq t)$ is a non-increasing function of t,)then the optimal price sequence is inherently non-decreasing.

Known distribution over n: Unbounded support

$v^*(t)$ gives for each state, an expected revenue for the seller, conditional on state t having been reached. The optimal pricing policy must satisfy the Bellman equation. So,

$$v^*(t) = max_{p_t}\{(1 - F(p_t))p_t + F(p_t)(1 - \phi(t))v^*(t + 1)\}$$

By expanding this formula and collecting the terms , we can convert this recursive formula into a series formulation that shows that $v^*(\tau) \leq v^*(\tau + 1)$. By contradiction on the optimality of p_{t+1},we can show that $p_t \leq p_{t+1}$. Therefore, the price p_t is also non-decreasing over time.

Known distribution over n: Unbounded support

If the seller has multiple units to sell, we include the number of remaining items(u) in the state. For any fixed u it still holds that if n is drawn from a distribution with *non-increasing hazard rate*, then the optimal price sequence is inherently non-decreasing. When a unit is sold, u is decremented. With less items to sell, and an increasing expected revenue, the price naturally doesn't decrease.

Problem Definition

- A sequence of independent non-negative random variables $x_1, x_2, \ldots, x_n$
- (Design a stopping rule) Choose an index $\tau \geq 1$
- Objective is to maximize x_τ.
- Decision has to made online.

Problem Definition Contd.

- Prophet Inequality measures the "goodness" of stopping rule designed by a gambler based only on past and present knowledge.
- Compared to a prophet's stopping rule who knows past, present and the future.
- $M = E(\max_i x_i)$, expected value of the maximum.
- $V = \max_\tau(E(x_\tau))$ expected value of the best stopping rule.
- Prophet inequality relates M and V.
- $M \leq \alpha V$, Find the smallest α.

General Problem

- A sequence of independent non-negative random variables $x_1, x_2, \ldots, x_n$
- (Design k stopping rules) Choose k indices, $1 \leq \tau_1 < \tau_2 < \ldots < \tau_k$
- Objective is to maximize $(x_{\tau_1} + x_{\tau_2} + \ldots + x_{\tau_k})$.

General Problem Contd.

- $M_k = E$(sum of the k largest elements).
- V_k =maximum of $E(x_{\tau_1} + \ldots + x_{\tau_k})$ over all sequences of stopping rules $\tau_1, \ldots, \tau_k$.
- $M_k \leq \alpha_k V_k$, Find the smallest α.

Prophet Inequlaity for $k = 1$

- $M = E(\max_i x_i)$, expected value of the maximum.
- $V = \max_\tau(E(x_\tau))$ expected value of the best stopping rule.
- $M \leq \alpha V$, .
- (Krengel and Sucheston [1977]) $\alpha = 2$ i.e. $M \leq 2V$.
  - 2 is optimal

Prophet Inequlaity for $k = 1$, $M \leq 2V$

- $M \leq 2V$ Constant 2 is optimal.
- $x_1 = 1, \quad x_2 = \begin{cases} L & \text{w.p } 1/L \\ 0 & \text{else} \end{cases}, \quad x_3 = \ldots = x_n = 0$
- $V = 1$ (τ is 1 or 2), since the expected value of all others is 0.
- $M = E(max\{1, x_2\}) = 1 \cdot (1 - 1/L) + L(1/L) = 2 - 1/L$
- Hence optimal ratio is 2.

Prophet Inequlaity for $k = 1$, $M \leq 2V$

- $M \leq 2V$ is achieved by one of these two stopping rules
- $M_n = \max_{i=1}^{n} x_i$,
  - $M = E(M_n)$
- Let m be the median of r.v M_n,
  - $Pr(M_n < m) \leq 1/2$ and $Pr(M_n > m) \leq 1/2$
- Stopping Rules
  - τ_1: Choose the first element that is $> m$. (else choose n)
  - τ_2: Choose the first element that is $\geq m$. (else choose n)
- Either $E(x_{\tau_1}) \geq E(M_n)$ or $E(x_{\tau_2}) \geq E(M_n)$

Prophet Inequlaity for $k = 1$, $M \leq 2V$

Let $(Z)^+ = \max\{0, Z\}$

Let $\beta = \sum_{i=1}^{n}(X_i - m)^+$ (sum of deviation from the median)

$E(M_n - m)^+ \leq \beta$, (any deviation is less than total deviation)

Prophet Inequlaity for $k = 1$, $M \leq 2V$

$$
\begin{aligned}
E(x_{\tau_1} - m)^+ &= E(\sum_{i=1}^{n}(x_i - m)^+ I(\tau_1 = i)) \\
&= E(\sum_{i=1}^{n}(x_i - m)^+ I(\tau_1 > i - 1)) \\
&= \sum_{i=1}^{n} E(x_i - m)^+ Pr(\tau_1 > i - 1)
\end{aligned}
$$

Since $Pr(\tau_1 > i - 1) \geq Pr(M_n \leq m) \geq 1/2$

$E(x_{\tau_1} - m)^+ \geq \beta/2$

- M deviates from the median by at most β
- $E(x_{\tau_1})$ deviates from the median by at least $\beta/2$

Single Item Online Auction via Prophet Inequality

- Item can be sold via posted price mechansim, price $= m$.
- 2 competitive
- Temporal strategy proof, bids are random variables.

Prophet Inequality: General Problem

- A sequence of independent non-negative random variables $x_1, x_2, \ldots, x_n$
- (Design k stopping rules) Choose k indices, $1 \leq \tau_1 < \tau_2 < \ldots < \tau_k$
- $M_k = E$(sum of the k largest elements).
- $V_k =$maximum of $E(x_{\tau_1} + \ldots + x_{\tau_k})$ over all sequences of stopping rules $\tau_1, \ldots, \tau_k$.
- $M_k \leq \alpha_k V_k$, Find the smallest α.

Prophet Inequality: Bounds on α_k

Hajiaghayi, Kleinberg, Sandholm 2007

- $M_k \leq \alpha_k V_k$, Find the smallest α_k.

$$1 + \sqrt{\frac{1}{512k}} \leq \alpha_k \leq 1 + \sqrt{\frac{8 ln(k)}{k}}$$

- for sufficiently large k

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

Given $x_1, x_2, \ldots, x_n,$ (≥ 0 and independent)
$\exists$ a constant m_k,
k stopping rules $\tau_1, \tau_2, \ldots, \tau_k$
indices of first k numbers $\geq m_k$ (if any)

$$cE(x_{\tau_1} + \ldots + x_{\tau_k}) \geq M_k).$$

where $c \geq 1 + \sqrt{\frac{8ln(k)}{k}}$

The upper bound: $\alpha_k \leq 1 + \sqrt{8 ln(k)/k}$

m_k: is the smallest number with expected number of x_i's exceeding it is around k.

$$\sum_{i=1}^{n} Pr(x_i > m_k) \leq k - \sqrt{2k\ln(k)}$$

$$\sum_{i=1}^{n} Pr(x_i \geq m_k) \geq k - \sqrt{2k\ln(k)}$$

Let $a_1 \geq a_2 \geq \ldots \geq a_k$ be the k largest values in the sequence.
So $M_k = OPT_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} a_i$

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

- Let $Z(r)$ be the number of elements $> r$ among the top k elements $\{a_1, \ldots, a_k\}$
- Let $W(r)$ be the number of elements $> r$ among the selected elements $\{x_{\tau_1}, \ldots, x_{\tau_k}\}$

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

$$OPT_k = a_1 + a_2 + \ldots + a_k \; = ka_k + \sum_{j=1}^{k-1} j(a_j - a_{j+1})$$

$$= \int_0^{a_k} k\,dr + \sum_{j=1}^{k-1} \int_{a_{j+1}}^{a_j} j\,dr$$

$$= \int_0^{a_k} k\,dr + \int_{a_k}^{a_{k-1}} k-1\,dr + \int_{a_{k-1}}^{a_{k-2}} k-2\,dr + \ldots + \int_{a_2}^{a_1} dr$$

$$= \int_0^{\infty} Z(r)\,dr$$

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

$$E(OPT_k) = M_k = \int_0^\infty E(Z(r))\, dr$$

Similarly,

$$E(x_{\tau_1} + \ldots + x_{\tau_k}) = \int_0^\infty E(W(r))\, dr.$$

We prove that $cE(W(r)) \geq E(Z(r))$ for $r \geq 0$.

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

Case 1: $r \geq a$**.** In this case by definition of $Z(r)$,

$$E(Z(r)) \leq \sum_{i=1}^{n} Pr(x_i > r)$$

And

$$E(W(r)) = \sum_{i=1}^{n} Pr(x_i > r \wedge i \in \{\tau_1, \ldots, \tau_k\})$$

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

It is sufficient to prove that

$$cPr(x_i > r \wedge i \in \{\tau_1, \ldots, \tau_k\}) \geq Pr(x_i > r)$$

or equivalently that

$$Pr(i \in \{\tau_1, \ldots, \tau_k\} | x_i > r) \geq \frac{1}{c}$$

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

Since $x_i > r \geq m_k$,

$$
\begin{aligned}
Pr(i \in \{\tau_1, \ldots, \tau_k\} | x_i > r) & \\
& \geq Pr\{\text{fewer than } k \text{ elements are } > m_k\} \\
& \geq \frac{1}{c}
\end{aligned}
$$

By Chernoff bound

The upper bound: $\alpha_k \leq 1 + \sqrt{8ln(k)/k}$

Case 2: $r < a$. Nothe that, $Z(r) \leq k$ and $W(r) > W(a)$.

$$
\begin{aligned}
E(W(r)) &> k - \sqrt{2l \ln k} \\
(1 + \sqrt{\frac{8ln(k)}{k}})E(W(r)) &> (1 + \sqrt{\frac{8ln(k)}{k}})(k - \sqrt{2l \ln k}) \\
&> k \\
&> E(Z(r)) \\
&> E(OPT_k)
\end{aligned}
$$

The Lower bound: $\alpha_k \geq 1 + \sqrt{1/512k}$

Theorem

For an integer $k \geq 8$, let$l = \lfloor \sqrt{k/8} \rfloor b$, let $n = 2k + l$; and let $x_1, x_2 \ldots x_n$ be independent random variables such that for $1 \leq i \leq l, x_i = 1$ with probability 1, and for $l < i \leq n, x_i = 0$ with probability $1/2$ and $x_i = 2$ with probability $1/2$. For any sequence of stopping rules $\tau_1 < \tau_2 < \ldots < \tau_k$, and any constant $c < (1 + \sqrt{1/512k})$, we have

$$cE(x_{\tau_1} + \ldots + x_{\tau_k}) < E(OPT_k(x_1, \ldots, x_n)).$$

The Lower bound: $\alpha_k \geq 1 + \sqrt{1/512k}$

Lemma: *If a fair coin is tossed $2k$ times, the probability that at least $k + \sqrt{k/8}$ of the tosses result in heads is $\geq 1/4$*

- Deviation of $1/2$ st-dev is very likely.

The Lower bound: $\alpha_k \geq 1 + \sqrt{1/512k}$

Let $\tau_1 < \tau_2 < \ldots < \tau_k$, be a sequence of k stopping rules
Let z be the number of 2's in the sequence $x_{l+1}, x_{l+2} \ldots x_n$.

$$Pr(z \geq k) > 1/2$$

and, By the above Lemma

$$Pr(z \leq k - l) \geq 1/4$$

The Lower bound: $\alpha_k \geq 1 + \sqrt{1/512k}$

$E(x_{\tau_1} + \ldots + x_{\tau_k}) \leq E(OPT_k) - E(m)$
m is the number of mistakes
Two types of mistakes

- Choosing an items among the first l elements when $z \geq k$
- Not choosing an item among the first l elements when $z \leq k - l$

So $Pr(\text{mistake at item } i) \geq 1/4$, Hence $m \geq l/4$
$E(x_{\tau_1} + \ldots + x_{\tau_k})(1 + \sqrt{1/512k}) < E(OPT_k)$

Unknown n and dependent bids

- $(x_1, x_2, ...)$, known joint distribution, uniformly bounded.
- Unkown n, known distribution.

For $k = 1$: $E(x_\tau) + 1/e \geq OPT_1(x_1, \ldots)$ (Hill & kertz 1983)

For $k > 1$ simple additive $k/2$ approximation:

$E(x_{\tau_1} + \ldots + x_{\tau_k}) + k/2 \geq E(OPT_k(x_1, \ldots, x_n))$

$O(k)$ additive error term can not be improved.

Unknown n and dependent bids

Given $(x_1, x_2, \ldots)$, $\quad x_i \in [0, 1]$.

Choose first k items $\geq 1/2$ (if any)

- gives $k/2$ appoximation

So $E(x_{\tau_1} + x_{\tau_2} + ... + x_{\tau_k}) + k/2 \geq E(OPT_k(x_1, x_2, ..., x_n))$

Unknown n and dependent bids: Optimality of $O(k)$

$$E(x_{\tau_1} + \ldots + x_{\tau_k}) + k/8 - 1 \leq E(OPT_k(x_1, \ldots, x_n))$$

- n is either k or k^3 equally likely
- $x_i = \begin{cases} 1 & \text{w.p } 1/k \\ 1/2 & \text{w.p } 1 - 1/k \end{cases}$

It achieves the bound

Unknown n and dependent bids

Let s be the number of items selected among the first k.
Let l be the number of ones among the first k

So $s = |\{1, \ldots, k\} \cap \{x_{\tau_1}, \ldots, x_{\tau_k}\}|$

and $l = |\{i : 1 \leq i \leq k \wedge x_i = 1\}|$

Case 1: $E(s) \leq k/2$
Case 2: $E(s) > k/2$

Unknown n and dependent bids: Case 1: $E(s) \leq k/2$

s is the number of items selected among the first k.
l is the number of ones among the first k

If $n = k$ The algorithm achieves:
$(x_{\tau_1} + \ldots + x_{\tau_k}) \leq l + s/2$

$E(x_{\tau_1} + \ldots + x_{\tau_k} | n = k) \leq E(l + s/2 | n = k) \leq 1 + k/4$

Since, $E(l) = 1$

$E(OPT | n = k) \geq k/2$

Unknown n and dependent bids: Case 1: $E(s) > k/2$

If $n = k^3$

$x_{\tau_1} + \ldots + x_{\tau_k} \leq l + s/2 + (k - s) = k + l - s/2$

$E(x_{\tau_1} + x_{\tau_2} + ... + x_{\tau_k} | n = k^3) \leq E(k + l - s/2 | n = k^3)$

If $n = k^3$

E(number of ones) $\geq k - 1$, w.h.p

So $E(OPT_k) \geq k - 1$