In the unique equib, all players follow this strategy, and all of them incur one unit of cost.

Assume that the objective function is to minimize the average cost incurred by players. In the above equi, this average cost is 1.

However splitting the traffic equally between the two links is the optimal solution with delay $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$. Thus $PoS = PoA = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

General selfish routing games are conceptually similar to Pigou's example, but more complex in several respects (arbitrary large directed graphs, different source-sink pairs, edge cost functions can be arbitrary nonnegative, continuous and non-decreasing functions). However even in the general case $PoA = PoS$.

We show that $\frac{4}{3}$ is indeed the PoA for all non-atomic selfish routings.

In general, we have a directed network together with a set $(s_1,t_1), \ldots, (s_k,t_k)$ of source-target pairs, called commodities. Each player is identified with one commodity. We use P_i to denote the s_i-t_i paths of a network and $P = \bigcup_{i=1}^{k} P_i$. We allow parallel edges, and a vertex can participate in multiple source-sink pairs.

For a flow f and a path $P \in P_i$, we interpret f_P as the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i.

There is demand d_i that needs to be satisfied, i.e., $\sum_{P \in P_i} f_P = d_i$

There is a cost function $c_e: R^+ \to R^+$ which is non-negative, continuous, and non-decreasing

A non-atomic instance is (G, d, C) → cost

selfish routing network demand

Let the cost of a path P with respect to a flow f as the sum of the costs of the constituent edges: $C_P(f) = \sum_{e \in P} c_e(f_e)$ where $f_e = \sum_{P \in P: e \in P} f_P$ (the flow on e)

Def: Nonatomic equilibrium flow: let f be a feasible flow for the nonatomic instance (G, d, c). The flow f is an equi flow, if for every commodity $i \in \{1, 2, \ldots, k\}$ and every pair $P, \tilde{P} \in P_i$ of s_i-t_i paths with $f_P > 0$, $C_P(f) \le C_{\tilde{P}}(f)$.

In other words, all paths in use by an equi. flow f have minimum-possible cost (see Pigou's example in which only one path carries flow)