We consider the objective of minimizing the total cost incurred by traffic. We define the cost of a flow f as $C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P = \sum_{e \in E} c_e(f_e) f(e)$

An optimal solution is the one minimizes the cost over all feasible flows.

- (In the atomic version the problem, each commodity represents a single player who must route a significant amount of traffic on a single path, where in the non-atomic case each commodity represents a large population of individuals, each of whom controls a negligible amount of traffic)

First we have the following theorem whose proof is omitted here but uses the potential function method (see section 18.3.1 of Algorithmic Game theory book)

- Theorem (Existence and uniqueness of equilibrium flows) See the proof in Page B

Let (G, d, c) be a nonatomic instance.

a) the instance (G, d, c) admits at least one equilibrium flow.

b) If f and $\tilde{f}$ are equa flows for (G, d, c) then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e.

- Now we prove the $\frac{4}{3}$ bound. Note that since x is continuous, the social optimum exists and well-defined.

First note that if a strateg f^{EQ} is an equilibrium then

$$\langle c(f^{EQ}), f^{EQ} - f \rangle \leq 0 \text{ for all strateg distribution (flow) } f \quad (1)$$

This is a direct consequence of the fact that in equi users travel on shortest paths with respect to $c(f^{EQ})$. Idea of the proof: (match the flow paths in f^{EQ} & f) if it is not the case then there is a negative cycle in $f^{EQ} - f$ and we can use of that cycle to find a shorter path. [Exercise: make it formalize] we consider each f^{EQ}_e & f_e together.

* Now Assume cost function $c_e(x) = c_e x + b_e$ for non-negative c_e, b_e for all edges (affine cost functions and separable then)

- Thm: POA is $\frac{4}{3}$.

Let f be the optimum social flow. Then by (1)

$$C(f^{EQ}) = \sum_e c_e(f^{EQ}_e) f^{EQ}_e \leq \sum_e c_e(f^{EQ}_e) f_e = \sum_e c_e(f_e) f_e + \sum_e \left(c_e(f^{EQ}_e) - c_e(f_e)\right) f_e$$