Proof of Thm (Existence and uniqueness of equilibrium flows)

Let (G, r, c) be a nonatomic instance where $c_e : \mathbb{R}^+ \to \mathbb{R}^+$ are nonnegative, continuous and non-decreasing.

a) The instance (G, r, c) admits at least one equilibrium flow

b) If f and $\tilde{f}$ are equilibrium flows for (G, r, c) then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge (e).

First, see this intuitive proposition from convex optimization

Proposition 1 (characterization of optimal flows) let (G, r, c^*) be a nonatomic instance such that for every edge e, the function $c^*_{(e)}$ is convex and continuously differentiable. Let $\dot{c}_e$ (derivative of c_e) denote the marginal cost function of the edge e. Then f^* is an optimal flow for (G, r, c) if and only if, for every commodity $i \in \{1, 2, \dots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ path with $f^*_P > 0$ $\dot{c}_P(f) \leq \dot{c}_{\tilde{P}}$

Now by setting $c^*_e(x) = \int_0^x c_e(y)\, dy$ for each edge e, we obtain the desired potential function. Moreover, since c_e is continuous and non decreasing for every edge e, every function c^*_e is both continuously differentiable and convex.

$$\text{call } \Phi(f) = \sum_{e \in E} c^*_e(f_e) = \sum_{e \in E} \int_0^{f_e} c_e(x)\, dx. \qquad (\star\star)$$

By the definition of nonatomic equil. flow (Def A), we have the following characterization of equil. flows as the global minimizers of the potential function Φ.

Prop 2 (potential function for equil. flows). Let (G, r, c) be a nonatomic instance. A flow feasible for (G, r, c) is an equil. flow if and only if it is a global minimum of the corresponding potential function Φ given in $(\star\star)$ above

-Now Proof of Thm: by definition the set of feasible flows of (G, r, c) can be identified with a compact (i.e., closed and bounded) subset of $|P|$-dim Eucl. space ($|P|$: # of pairs). Since edge cost functions are continuous, the potential function is a continuous function on this set. By Weierstrass' theorem from elementary math analysis, the potential function Φ achieves a minimum value on this set. By Prop 2., every point at which Φ attains its minimum corresponds to an equi flow. Part (b) follows essentially because of convexity of Φ, which itself follows from the fact that each cost function is non-decreasing, and hence each summand on $(\star\star)$ is convex. (we take two equi flow $f, \tilde{f}$ which by prob 2 minimize Φ as well) and then it is not hard to see that c_e is constant between f_e and $\tilde{f}_e$. See the exact details in Page 470 of AGT book)