using Karush, Kuhn, Tucker (KKT) conditions which are essentially the generalization of duality theory for linear programming, we know that optimal solutions (an complementarity slackness conditions) to x_{ij} and p_j must satisfy the following: (here p_j's are dual variables corresponding to second set of conditions)

i) $\forall j \in G; p_j \geq 0$

ii) $\forall j \in G: p_j > 0 \Rightarrow \sum_{i \in B} x_{ij} = 1$

iii) $\forall i \in B, \forall j \in G: \frac{u_{ij}}{p_j} \leq \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$

iv) $\forall i \in B, \forall j \in G: x_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i}$

Thm: For the linear case of Fisher's Model:

a) If each Good has a potential buyer, equil exists.

b) The set of equil allocations is convex

c) Equil utilities and prices are unique.

Proof: since for good j, there is buyer i with $u_{ij} > 0$, by (iii) $p_j \geq \frac{B_i u_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} > 0$.

Thus by (ii), $\sum_{i \in B} x_{ij} = 1$ and are goods are fully sold their prices are positive. also

iii) an iv) imply that if buyer i gets good j (i.e. $x_{ij} > 0$) then j must be among the goods that give buyer i maximum utility per unit money spent at current prices. Hence each buyer gets only a bundle consisting of her most desired goods, i.e., an optimal bundle.

By (iv), $\frac{u_{ij}}{p_j} \times x_{ij} = \frac{\sum_{j \in G} u_{ij} x_{ij}}{B_i} \times x_{ij}$ to handle the case of $x_{ij} = 0$ $\forall i \in B, j \in G$.

$\Rightarrow \frac{B_i u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = p_j x_{ij}$

sum over all j: $\frac{B_i \sum_{j \in G} u_{ij} x_{ij}}{\sum_{j \in G} u_{ij} x_{ij}} = \sum_{j \in G} p_j x_{ij} \Longrightarrow \forall i \in B: B_i = \sum_j p_j x_{ij}$ and thus

money of each buyer is fully spent. This complete the proof of (a).

- Since each equil allocation is an optimal solution to the Eisenberg-Gale
- Convex Program, the set of allocations must form a convex set. Thus we have (b).
- Since log is strictly concave, similar to non-atomic selfish routing, if there is more than one equil, the utility by each buyer should be the same and this with (iv) gives unique equil. Prices. Thus we have (c) also.