Def: A (direct revelation) ~~Au~~ Auction (A.K.A mechanism) is a social choice function $f : V_1 \times \cdots \times V_n \to A$ and a vector of payment functions $p_1, \ldots, p_n$

where $p_i : \underbrace{V_1 \times \cdots \times V_n}_{\text{or } A \text{ (in several cases)}} \to \mathbb{R}$ is the Amount that player i pays.

(A: action or alternative)

Def: A mechanism $(f, p_1, p_2, \ldots, p_n)$ is called $\overline{\text{strategy-proof or truthful}}$ incentive compatible if for every player i, every $v_1 \in V_1, \ldots, v_n \in V_n$ and every $v_i' \in V_i$, if we denote $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$ then

$$v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v_i', v_{-i}).$$

It means player i prefer "telling the truth" v_i to the mechanism rather than any possible "lie" v_i', since this gives him higher (in the weak sense) utility.

when there is money, there is an incentive compatible mechanism for the most natural social choice function: optimizing the social welfare (or efficiency), which is for an alternative $a \in A$ is the sum of valuations of all players for this alternative, $\sum_i v_i(a)$.

Def: A mechanism $(f, p_1, \ldots, p_n)$ is called a Vickrey-Clarke-Groves (VCG) mechanism if

- $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum v_i(a)$; that is f maximizes the social welfare

and

- for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \to \mathbb{R}$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$,

$$p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \underbrace{\sum_{j \neq i} v_j(f(v_1, \ldots, v_n))}$$

each player is paid an amount equal to the sum of the values of all other players. when we add this to $v_i(f(v_1, \ldots, v_n))$ the sum becomes exactly the total social welfare of $f(v_1, \ldots, v_n)$.