CS323, Spring 2009
Homework #2 (Solutions)

Problem I: **(i)** $f(1) = -4$ and $f(2) = 5$.

Hence there exists a c between $[1, 2]$ such that $f(c) = 0$, and the equation has one root in this interval.

Now consider $f'(x) = 3x^2 + 2 > 0,\ \forall\ x \in \mathbb{R}$ Hence $f(x)$ is an increasing function and crosses the x-axis only once and the crossing point lies in the interval $[1, 2]$.

(ii)

$$c_1 = 1.5, \quad f(c_1) = x_0 = -0.625$$

$$c_2 = 1.75, \quad f(c_2) = x_1 = 1.86$$

$$c_3 = 1.625, \quad f(c_3) = x_2 = 0.541$$

Error bound at x_2 is:

$$|e_2| \le \frac{2-1}{2^{2+1}} = \frac{1}{8} = 0.125$$

(iii)

$$|e_k| \le \frac{2-1}{2^{k+1}} \le 10^{-5}$$

$$\Longrightarrow\ k \ge 17$$

Problem II: **(i)** (a)$g'(x) = -\frac{3x^2}{2}$ So: $\max_{[1,2]}|g'(x)| = \frac{3}{2} > 1$. Thus $|g'(\bar{x})| > 1$ Hence this iteration, doesn't converge.

(b) $g'(x) = -\frac{2}{3} \cdot (7 - 2x)^{-\frac{2}{3}}$, and $\max_{[1,2]}|g'(x)| = \frac{2}{3} \cdot 3^{-\frac{2}{3}} < 1$ Thus $|g'(\bar{x})| < 1$, and hence converges.

(c) $g'(x) = \frac{1}{7}(4x^3 + 4x)$ and $\max_{[1,2]}|g'(x)| = \frac{1}{7} \cdot 40 > 1$, hence$|g'(\bar{x})| > 1$, and the iteration doesn't converge.

(d) $g'(x) = \frac{6x^4+12x^2-42x}{(3x^2+2)^2}$, and similarly, $\max_{[1,2]}|g'(x)| = 0.96 < 1$, hence $|g'(\bar{x})| < 1$ and the iteration is locally convergent.

(iii) Using (b), after 7 iterations we get $x = 1.5690$

Problem III $g(x) = \frac{1}{2}e^{-x}$. For the interval $x \in [0, 1]$, $\quad g(x) \in [0, 1]$. Therefore the equation $x = g(x)$ has at least one solution in $[0, 1]$. Now, $\max_{0 \le x \le 1}|g'(x)| < \frac{1}{2} < 1$. Therefore, by CMT, there is a unique solution of $x = g(x)$ in the interval $[0, 1]$ and for any initial value $x_0 \in [0, 1]$, the fixed point iteration will converge to the solution.

As $x_k \to \bar{x}$, $e_{k+1} \approx g'(\bar{x}) \cdot e_k$, and $|g'(\bar{x})| < \frac{1}{2}$. The bisection method, reduces the error factor by a $\frac{1}{2}$ each time, so convergence rate of fixed-point method is slightly faster than the bisection method.

Problem IV The iterations stop when $|x_{k+1} - x_k| < 10^{-3}$

(i) $x = 2.2242798$,

(ii) $x = 2.22375$

Problem V We are given that the range of interest is $0 < |g'(\bar{x})| < 1$.

For the stopping condition to be “safe”, the following inequality has to be met:

$$|x_{k+1} - \bar{x}| \leq |x_{k+1} - x_k|$$

Now we have:

$$x_{k+1} - \bar{x} \approx \frac{-g'(\bar{x})}{1 - g'(\bar{x})}(x_{k+1} - x_k)$$

So the “safe” condition will only be met when

$$\left|\frac{-g'(\bar{x})}{1 - g'(\bar{x})}\right| < 1$$

Safe values are $(-1 < g'(\bar{x}) < 0) \;\cup\; (0 < g'(\bar{x}) < \frac{1}{2})$

Problem VI