Problem I: **(i)** We have:

$$\begin{array}{c|cccccc} & 1 & 0 & 0 & -4 & 7 & 6 \\ 2 & & 2 & 4 & 8 & 8 & 30 \\ \hline & 1 & 2 & 4 & 4 & 15 & 36 \end{array}$$

So $p(2) = b_{n+1} = 36$.
Now from above we have $q(x) = x^4 + 2x^3 + 4x^2 + +4x + 15$, and we know that $p'(2) = q(2)$
So we have:

$$\begin{array}{c|ccccc} & 1 & 2 & 4 & 4 & 15 \\ 2 & & 2 & 8 & 24 & 56 \\ \hline & 1 & 4 & 12 & 28 & 71 \end{array}$$

(ii) We know that:

$$p(x) = (x - z)q(x) + b_0$$

Differentiating twice, we have:

$$p'(x) = (x - z)q'(x) + q(x)$$

$$p''(x) = (x - z)q''(x) + 2q'(x))$$

Evaluating at $x = z$:

$$p''(z) = 2q'(z)$$

We can find $q'(x)$ by another application of synthetic division to part(i)

$$\begin{array}{c|cccc} & 1 & 4 & 12 & 28 \\ 2 & & 2 & 12 & 48 \\ \hline & 1 & 6 & 24 & 76 \end{array}$$

So, $q'(2) = 76$, and $p''(2) = 2 \cdot q'(2) = 152$

Problem II: **(i)**

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 2 & 3 & 7 \end{array}\right]$$

Eliminating x_1 from R_3 by $R_3 \leftarrow R_3 - 2R_1$, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 7 \end{array}\right]$$

Eliminating x_2 from R_3 by $R_3 \leftarrow R_3 - 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

By backward elimination:

$$3x_3 = 3, \quad x_3 = 1$$

$$x_2 - x_3 = 1, \quad x_2 = 2$$

$$x_1 + x_3 = 1, \quad x_3 = 1$$

(ii)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

$$x = A^{-1}b$$

and

$$AA^{-1} = I$$

We have the superaugmented matrix,to perform Gaussian elimination on:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right] \ldots$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{array}\right]$$

$$\downarrow_{\frac{1}{3}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{-2}{3} & \frac{1}{3} \end{array}\right]$$

$$\downarrow_{R_2 \leftarrow R_1 - R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{-2}{3} & \frac{1}{3} \end{array}\right]$$

$$\downarrow_{R_2 \leftarrow R_2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{-2}{3} & \frac{1}{3} \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Since $A^{-1}b = x$

$$x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(iii) From Gaussian elimination we have the following matrix, where mutlipliers are within brackets:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ (2) & (2) & 3 \end{bmatrix}$$

So, we have:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Solving $Ly = b$, we have:

$$y = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Solving $Ux = y$:

$$x = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Problem III

$$\begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow_{R_1 \leftrightarrow R_2}$$

$$\begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 1/2R_1} \begin{bmatrix} -2 & 2 & 0 \\ (\frac{1}{2}) & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + \frac{1}{2}R_1} \begin{bmatrix} -2 & 2 & 0 \\ (\frac{1}{2}) & 0 & 2 \\ (-\frac{1}{2}) & 1 & 1 \end{bmatrix}$$

$$\downarrow_{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} -2 & 2 & 0 \\ (-\frac{1}{2}) & 1 & 1 \\ (\frac{1}{2}) & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} U = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2 \text{ and } R_2 \leftrightarrow R_3} b' = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

Solving $Ly = b'$:

$$y = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

Solving $Ux = y$:

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Problem IV (i)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 2 & 3 & 7 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_2 + \frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_1 - \frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(ii) We will count only multiplications/divisions. When introducing zeros in the first column, we do $(n+1)$ multiplications for each of the $n-1$ rows, so we have $(n+1)\cdot(n-1)$ operations for the first step. Similarly the second step involves n multiplications on each of the $n-1$ rows to get zeros in the second column. At the nth step there are $n-1$ rows and 3 multiplications for each row. At the end the diagonal matrix needs n divisions to find the solution.

So total number of operations is :

$$(n-1)\sum_{i=3}^{n+1} i + n$$

$$= \frac{n^3}{2} + n^2 - \frac{5n}{2} + 2$$

$$\approx \frac{n^3}{2}$$

Problem V **(i)**

$$A = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Using $R_3 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 - 2R_2$ and $R_4 \leftarrow R_4 - R_3$, we have:

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ (-1) & 1 & 2 & 0 \\ 0 & (2) & -1 & -1 \\ 0 & 0 & (1) & 3 \end{bmatrix}$$

$$L \cdot y = b$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot y = \begin{bmatrix} 6 \\ 1 \\ 3 \\ 3 \end{bmatrix}$$

$$\implies y = \begin{bmatrix} 6 \\ 7 \\ -11 \\ 14 \end{bmatrix}$$

$$U \cdot x = y$$

$$\implies \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \cdot x = \begin{bmatrix} 6 \\ 7 \\ -11 \\ 14 \end{bmatrix}$$

$$\implies x = \begin{bmatrix} -\frac{8}{3} \\ -\frac{17}{3} \\ \frac{19}{3} \\ \frac{14}{3} \end{bmatrix}$$

(ii) For each step, while eliminating a variable from, we perform only two multiplications per row, since only two terms in a row are non-zero. So total no, of multiplications/divisions for step(a) is $2(n-1)$.

Consider the system $Ly = b$. Each of the$n-1$ rows after the first row in L has 2 non-zero entries, one of which we know to be 1. So total no. of multiplications/divisions for solving this system is $(n-1)$. For $Ux = y$, $n-1$ rows have two non-zero entries, hence total no. of operations to solve this system is $2(n-1)+1$. Therefore total no. of operations to solve the LU system is $3n-2$

Problem VI Data:$(0,1), (\frac{\pi}{2},1), (\pi,0), (\frac{3\pi}{2},2)$

$$y(x) = c_1 + c_2 \cos x + c_3 \sin x$$

So:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$R \cdot c = y$$

$$\therefore R^T \cdot R \cdot c = R^T \cdot y$$

$$\implies \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \; c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

$$\implies c_1 = 1, \; c_2 = \frac{1}{2}, \; c_3 = -\frac{1}{2}$$

The least square approximation is:

$$y(x) = 1 + \frac{1}{2}\cos x - \frac{1}{2}\sin x$$