Homework 4 Solutions

Darakhshan Mir

May 3, 2009

I (i) **Lagrange**

$$P_3(x) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 + L_3(x)y_3$$

where

$$L_0(x) = -\frac{x^3 - 6x^2 + 11x + 6}{6}$$
$$L_1(x) = \frac{x^3 - 5x^2 + 6x}{2}$$
$$L_2(x) = \frac{x^3 - 4x^2 + 3x}{2}$$
$$L_3(x) = \frac{x^3 - 3x^2 + 2x}{6}$$

and

$$P_3(x) = \frac{5}{6}x^3 - 4x^2 + \frac{25}{6}x - 1$$

Newton

$$P_3(x) = [0]+(x-0)[0,1]+(x-0)(x-1)[0,1,2]+(x-0)(x-1)(x-2)[0,1,2,3]$$

We have the table of divided differences:

$$[0,1] = 1, [1,2] = 2, [2,3] = 0$$
$$[0,1,2] = -\frac{3}{2}, [1,2,3] = 1$$
$$[0,1,2,3] = \frac{5}{6}$$

$$P_3(x) = \frac{5}{6}x^3 - 4x^2 + \frac{25}{6}x - 1$$

(ii)

$$f(\frac{1}{2}) \approx P_3(x) = \frac{75}{24}$$

and

$$\max_{x\in[0,3]} |f(\frac{1}{2}) - P_3(\frac{1}{2})| = \max_{x\in[0,3]} |(\frac{1}{2} - 0)(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)|$$

$$= \frac{\frac{15}{16} \cdot M_4}{4!} = \frac{15}{384} M_4$$

II (i)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

The augmneted matrix may be written as:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 & -1 \\ 0 & 3 & 9 & 27 & -1 \end{array}\right]$$

On solving we get $c_3 = \frac{5}{6}, c_2 = -4, c_1 = \frac{25}{6}, c_0 = 1$

So the interpolating polynomial is:

$$\frac{5}{6}x^3 - 4x^2 + \frac{25}{6}x + 1$$

(ii) Using Newton's interpolaton, we need $\frac{1}{2}n^2$ operations to construct the table of divided differences and $O(n)$ operations to compute the polynomial. Whereas Gaussian elimination without pivoting takes $\frac{1}{3}n^3 + O(n^2)$ operations

III (i)

$$P_1(x) = \frac{(x-8)}{(7-8)} \cdot 1.946 + \frac{(x-7)}{8-7)} \cdot 2.079$$

$$P_1(x) = 15.5680 - 1.946x + 2.079x - 14.5530$$

$$P_1(x) = 0.1330x + 1.0150$$

$$\implies P_1(7.2) = 1.9726)$$

The error

$$E_1(x) = |f(x) - P_1(x)| \le \frac{(x-7)(x-8)}{2} \max_{x\in[7,8]} \frac{-1}{x^2}$$

$$E_1(7.2) \le 0.0016327$$

(ii)

$$P_2(x) = \frac{(x-7)(x-8)}{(6-7)(6-8)} \cdot 1.792 + \frac{(x-6)(x-8)}{(7-6)(7-8)} \cdot 1.946 + \frac{(x-6)(x-7)}{(8-6)(8-7)} \cdot 2.079$$

$$= P_2(x) = -0.0105x^2 - 0.2905x + 0.427$$

$$P_2(7.2) = 1.9741$$

The error

$$E_2(x) = |f(x) - P_2(x)| \le \frac{(x-6)(x-7)(x-8)}{6} \max_{x\in[6,8]} \frac{2}{x^3}$$

$$E_2(7.2) \le 2.963 \times 10^{-4}$$

Hence $P_2(7.2) =$ is more accurate

IV (i) The following hold true for the error, for any point $a \in \{x_1 \dots x_{m-1}\}$:

$$E_1(x) \leq \max_{x \in [a,a+h]} [(x-a)(x-(a+h))] \frac{f^2(c)}{2} \; c \in [1,3]$$

$$E_1(x) \leq \frac{h^2}{8} \max_{x \in [1,3]} \frac{f^2(c)}{2} \; c \in [1,3]$$

$$\implies E_1(x) \leq \frac{h^2}{8} \leq 10^6$$

$$h \leq 0.0028284$$

We have $n.h = 3 - 1$ So, $n \geq 708$

(ii) Similarly

$$E_2(x) \leq \max_{x \in [a,a+2h]} [(x-a)(x-(a+h))(x-(a+2h))] \frac{f^3(c)}{6} \; c \in [1,3]$$

$$E_2(x) \leq \frac{2\sqrt{3}}{9}.h^3. \max_{c \in [1,3]} \frac{2}{6c^3} \leq 10^{-6}$$

So,

$$nh = 3 - 1 \; and n \geq 152$$

VI (ii) add

(a) We will use equation (1).
We have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and $\bar{A} = A$ Using $\bar{\bar{A}} = \alpha \bar{A} + \frac{1-\alpha}{n} ee^T$, we have

$$\bar{\bar{A}} = \begin{pmatrix} 0.025 & 0.875 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0.025 & 0.025 & 0.45 & 0.875 & 0.025 & 0.025 \\ 0.45 & 0.025 & 0.025 & 0.025 & 0.025 & 0.025 \\ 0.45 & 0.025 & 0.450 & 0.025 & 0.025 & 0.025 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.025 & 0.875 \\ 0.025 & 0.025 & 0.025 & 0.025 & 0.875 & 0.025 \end{pmatrix}$$

Consider $p^{(0)} = [0\;0\;0\;0\;0\;1]^T$ ans using equaton(1) we see that $p^{(k+1)}$ converges to

$$r = \begin{pmatrix} 0.19814 \\ 0.20369 \\ 0.10921 \\ 0.15562 \\ 0.16667 \\ 0.16667 \end{pmatrix}$$

Using equation(8) we have $(I - 0.85A)s = \frac{[1\ 1\ 1\ 1\ 1]^T}{6}$ Solving for s we get

$$s = \begin{pmatrix} 1.32093 \\ 1.35796 \\ 0.72806 \\ 1.03749 \\ 1.11111 \\ 1.11111 \end{pmatrix}$$

From equation(9) we have$\gamma = 1 - \alpha = 1 - 0.85$, since χ is a null matrix. So from (8) we have $r = 0.15 * s$, which gives

$$r = \begin{pmatrix} 0.19814 \\ 0.20369 \\ 0.10921 \\ 0.15562 \\ 0.16667 \\ 0.16667 \end{pmatrix}$$

(b) add