

Homework 5 Solutions

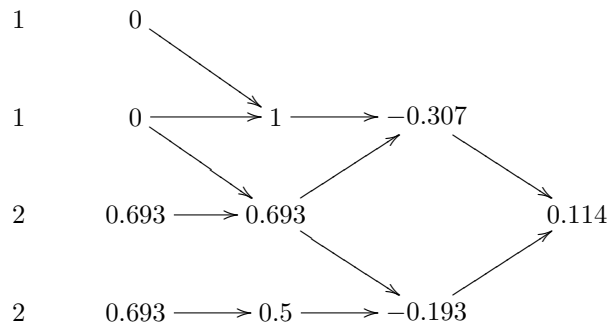
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I (i) We have

$$f(1) = 0, f(2) = 0, f'(1) = 1, f'(2) = 0.5$$

We have:



$$\begin{aligned} \Rightarrow P_3(x) &= 0 + 1(x-1) - 0.307(x-1)^2 + 0.114(x-1)^2(x-2) \\ P_3(1.2) &= 0.184702 \end{aligned}$$

(ii)

$$f(x) - P_3(x) = (x-1)^2(x-2)^2 \frac{f^4(c)}{4!} \quad c \in [1, 2]$$

$$\text{Now } \max_{c \in [1, 2]} |f^4(c)| = \max_{x \in [1, 2]} \left| \frac{-6}{x^4} \right| = 6$$

$$|f(1.2) - P_3(1.2)| \leq 0.04 \cdot 0.64 \cdot \frac{6}{24} = 0.0064$$

II (i) **Trapezoidal**

$$\begin{aligned} I_S^4 &= \frac{\pi}{12} \left[\sin 0 + 2 \sin \frac{\pi}{6} + 2 \sin \frac{\pi}{3} + 2 \sin \frac{\pi}{2} + \sin \frac{2\pi}{3} \right] \\ &= 1.4656 \end{aligned}$$

Simpson's

$$\begin{aligned} I_S^4 &= \frac{\pi}{18} \left[\sin 0 + 4 \sin \frac{\pi}{6} + 2 \sin \frac{\pi}{3} + 4 \sin \frac{\pi}{2} + \sin \frac{2\pi}{3} \right] \\ &= 1.5006 \end{aligned}$$

(ii)

$$\begin{aligned} E_{trap} &= \left| \frac{\frac{2\pi}{3} \left(\frac{\pi}{6}\right)^2}{12} \cdot \sin \frac{\pi}{2} \cdot f''(c) \right| c \in \left[0, \frac{2\pi}{3}\right] \\ \implies E_{trap} &\leq \left| \frac{\pi^3}{648} \max_{c \in [0, \frac{2\pi}{3}]} f''(c) \right| \\ &= 0.0478 * \sin \frac{\pi}{2} = 0.0478 \\ E_{Simp} &= \left| \frac{\frac{2\pi}{3} \cdot \left(\frac{\pi}{6}\right)^4}{180} f^{(4)}(c) \right| c \in \left[0, \frac{2\pi}{3}\right] \\ &= 8.7454 \times 10^{-4} \end{aligned}$$

III (i) Solve

$$\begin{aligned} \frac{4h^2}{12} \cdot |f''(c)| c \in [1, 5] &\leq 10^{-8} \\ f''(x) &= \frac{1}{x} \end{aligned}$$

and

$$\begin{aligned} n * h &= 4 \\ \therefore n &\geq 23094 \end{aligned}$$

(ii)

$$\begin{aligned} \frac{4h^4}{180} \cdot |f^{(4)}(c)| c \in [1, 5] &\leq 10^{-8} \\ \max_{x \in [1, 5]} f^{(4)}(x) &= \max_{x \in [1, 5]} \frac{2}{x^3} = 2 \\ h \leq 217.7 \quad n * h = 4 \quad n &\geq 184 \end{aligned}$$

IV (i) The formula is exact for $f(x) = 1, x, x^2, x^3$

$$\begin{aligned} \int_0^{2h} 1 \cdot dx &= 2h = w_0 \cdot 1 + w_1 \cdot 1 + w_2 \cdot 0 + w_3 \cdot 1 \\ \int_0^{2h} x \cdot dx &= 2h^2 = w_0 \cdot 0 + w_1 \cdot h + w_2 \cdot 1 + w_3 \cdot 2h \\ \int_0^{2h} x^2 \cdot dx &= \frac{8h^3}{3} = w_0 \cdot 0 + w_1 \cdot h^2 + w_2 \cdot 2h + w_3 \cdot 4h^2 \\ \int_0^{2h} x^3 \cdot dx &= 4h^4 = w_0 \cdot 0 + w_1 \cdot h^3 + w_2 \cdot 3h^2 + w_3 \cdot 8h^3 \end{aligned}$$

Solving we get

$$w_0 = \frac{h}{3}, w_1 = \frac{4h}{3}, w_2 = 0, w_3 = \frac{h}{3}$$

(ii) We have

$$|f(x) - P_3(x)| = (x-a)(x-(a+h))^2(x-(a+2h)) \frac{f^{(4)}(c)}{4!} \quad c \in [a, a+2h]$$

$$\begin{aligned} E_I &= \int_a^{a+2h} [f(x) - P_3(x)] dx \\ &= \int_a^{a+2h} (x-a)(x-(a+h))^2(x-(a+2h)) \cdot \chi dx \end{aligned}$$

where

$$\chi = \frac{f^{(4)}(c)}{4!} \quad c \in [a, a+2h]$$

Since all the other terms don't change sign over the interval χ can be taken out of the integral Also changing limits we have:

$$\begin{aligned} E_I &= \chi \int_0^{2h} t(t-h)^2(t-2h) dt \\ &= -\frac{h^5}{90} f^{(4)}(c) \quad c \in [a, a+2h] \\ &= -\frac{h^4 \cdot [(a+2h) - 2h]}{180} \cdot f^{(4)}(c) \quad c \in [a, a+2h] \end{aligned}$$

which is the error in Simpson's formula.

- V (i) Since the formula is based on linear interpolation it will be exact for $f(x) = 1, f(x) = x$

We have:

$$\begin{aligned} \int_{-h}^h 1 \cdot dx &= 2h = w_0 + w_1 \\ \int_{-h}^h x dx &= 0 = \frac{-h}{\sqrt{3}} w_0 + \frac{h}{\sqrt{3}} w_1 \end{aligned}$$

On solving, we have

$$w_0 = h, w_1 = h$$

- (ii) For $f(x) = x^2$

$$\int_{-h}^h hx^2 dx = \frac{2h^3}{3}$$

and the formula evaluates to

$$h \left[\left(-\frac{h}{\sqrt{3}}\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2 \right] = \frac{2h^3}{3}$$

Hence the is exact for x^2

For $f(x) = x^3$

$$\int_{-h}^h hx^3 dx = \frac{2h^5}{9}$$

and the formula evaluates to

$$h \left[\left(-\frac{h}{\sqrt{3}}\right)^4 + \left(\frac{h}{\sqrt{3}}\right)^4 \right] = \frac{2h^5}{9}$$

Hence the formula is exact for all cubics.

(iii) Applying the above Gaussian quadrature formula

$$I_G \approx h \left[\left(-\frac{h}{\sqrt{3}}\right)^4 + \left(\frac{h}{\sqrt{3}}\right)^4 \right] = \frac{2h^5}{9}$$

Applying Simpson's rule

$$I_S = \frac{h}{3} [(-h)^4 + 4.0 + h^4] = \frac{2h^5}{3}$$

The exact value

$$I = \int_{-h}^h x^4 dx = \frac{2h^5}{5}$$

So I_G gives a more accurate result than I_S .

V

$$f(x) = \sin x \quad a = 0, b = \frac{2\pi}{3}$$

For $n = 1, h = \frac{2\pi}{3}$

$$T_1 = \frac{b-a}{2} [f(a) + F(b)]$$

$$T_1 = \frac{2\pi}{3 \cdot 2} \left[\sin 0 + \sin \frac{2\pi}{3} \right] = 0.9069$$

For $n = 2, h = \frac{\pi}{3}$

$$T_2 = \frac{1}{2} T_1 + h f(a+h) = 1.3603$$

for $n = 4, h = \frac{\pi}{6}$

$$T_4 = \frac{1}{2} T_2 + h [f(a+h) + f(a+3h)] = 1.4656$$

Similarly we have:

$$T_2^* = T_2 + \frac{1}{3} [T_2 - T_1] = 1.5114$$

$$T_4^* = T_4 + \frac{1}{3} [T_4 - T_2] = 1.5007$$

And finally

$$T_4^{**} = T_4^* + \frac{1}{15} [T_4^* - T_2^*] = 1.4999$$