EXP : the missing equivalence

Sambuddha Roy*

1 Lowness of $AM \cap coAM$

The fact that we will repeatedly use in the following is the result of [Sch89] that $AM \cap coAM$ exactly capture the low sets for AM. We do not have a similar nice result for MA - if we run through the proof of lowness, we see that it is mainly because while only one Merlin-Arthur round gives MA, any bounded number (≥ 2) of Merlin-Arthur rounds gives AM.

Here we prove

Proposition 1 *For any* $\mathcal{C} \in \{EXP, NEXP, FewEXP, P^{\#P}, PSPACE, EXP^{NP}\}$*, we have*

$$\mathcal{C} \subset (AM \cap coAM)/poly$$
$$\Leftrightarrow$$
$$\mathcal{C} = AM$$

Proof: Taking an instance, let $\mathcal{C} = EXP$. We know that if $EXP \subset (AM \cap coAM)/poly$ then by relativizing [BFNW93] we have $EXP \subseteq MA^{AM \cap coAM}$ and this last class is contained in AM by the lowness result cited above. Therefore, $EXP = AM$.

For the other direction, if $EXP = AM$ then $AM = coAM$ and hence $EXP = (AM \cap coAM) \subset (AM \cap coAM)/poly$. Hence the equivalence holds.

For $\mathcal{C} = NEXP$, we first want to prove that $NEXP = EXP$. For this we apply [IKW02] and that $EXP = AM$ under these assumptions (by above):

If $NEXP \neq EXP$, $AM \subset io - NTime(2^{n^{\epsilon}})/n^{\epsilon}$, also, $NEXP \subset \Sigma_2/poly$, so this implies, $EXP \subset io - \Sigma_2 Size(n^c)$ for a fixed c, a contradiction, via [Kan82].

The other direction holds similarly to that for EXP. For $FewEXP$ we use the result in [AKS95] :

Fact 2 *If* $FewEXP \subset EXP/poly$ *then* $FewEXP = EXP$.

For EXP^{NP} we use the result in [BH92] :

Fact 3 *If* $EXP^{NP} \subset EXP/poly$ *then* $EXP^{NP} = EXP$.

Comment 1 *Our proof generalizes Vinodchandran's note, [Vin04].*

*Department of Computer Science, Rutgers University, Piscataway, NJ 08855

2 About EXP

Proposition 4 *The following equivalence holds for EXP :*

$$EXP \subset P/poly \\ \Leftrightarrow \\ EXP = IP[P/poly]$$

Proof: The proof follows readily from the literature ([BFNW93], [IKW02],$\cdots$). If $EXP \subset P/poly$, given the checkers for EXP we can see that it is in $IP[P/poly]$. On the other hand, $IP[P/poly]$ is a class slightly above BPP but still in $P/poly$ (cf. [AKS95]). So $EXP = IP[P/poly]$ implies $EXP \subset P/poly$ too.

$IP[P/poly]$ is an interesting class - it is only slightly above BPP and yet we do not know if it is closed under complementation - it is low for several classes, like ZPP^{NP}, S_2^p.

Altogether we have the following equivalences for EXP

$$EXP \subset P/poly \Leftrightarrow EXP = IP[P/poly] \\ EXP \subset (AM \cap coAM)/poly \Leftrightarrow EXP = AM$$

The equivalence missing corresponds to the situation squashed in between the two cases above. $EXP \subset \mathcal{C}/poly \Leftrightarrow EXP = MA$.
But at least from the above, we do know that if we want to prove something wrt EXP and MA we have to prove nonuniform lower bounds.

It is instructive to compare the above equivalences with the one due to [FK05]

$$EXP \subset NP/log \Leftrightarrow EXP = P_{\|}^{NP}$$

3 Discussion

[SU05] ask if $EXP = P_{\|}^{NP}$ implies $EXP = AM$ - from the above equivalences, we might have a better idea as to what kind of nonuniform downward collapses we are asking for.

[IKW02] show a result for $NEXP$ vs MA but we do not get one such for EXP vs MA. It is intriguing as to why not.

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