Homework 2 Solutions

Q#2. (pg.316) This is straightforward verification.

Q#14. (pg.317) For paying a bill of n pesons, the first (since the order of paying the coins and bills actually matters) coin/bill chosen with which to pay the bill off, might be any of 1 peso, 2 pesos, and so on. Let a_n be the number of ways of payment for a bill of n pesos. Suppose, the first partial payment made is of 1 peso. Then we still have to pay $(n-1)$ pesos, the payment of which can be made in a_{n-1} ways, likewise, if the first payment is of 2 pesos, the rest of the payment can be made in a_{n-2} ways, and so on. Altogether, collection all such ways of payment, based on the first partial payment, we have the recurrence for a_n :
$a_n = a_{n-1} + a_{n-2} + a_{n-5} + a_{n-10} + a_{n-5} + a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}$. (we can of course collect the two terms corresponding to a_{n-5} and a_{n-10})

Q#34. (pg.318) Let e_n denote the number of bit sequences of length n with an even number of 0's. Note that therefore there are $2^n - e_n$ bit sequences with an odd number of 0's. There are two ways to get a bit string of length n with an even number of 0's. Here, what we do is look at the ending bit as we discussed in the recitation. The last bit can be either a 1 and be preceded by a bit string of length $n-1$ with an even number of 0's, and by the definition (of e_n) there are e_{n-1} of these; or it can end with a 0 and be preceded by a bit string of length $n-1$ with an odd number of 0's, and there are $2^{n-1} - e_{n-1}$ of these. Altogether, therefore, $e_n = e_{n-1} + 2^{n-1} - e_{n-1}$, or simply, $e_n = 2^{n-1}$.

Q#6. (pg.350) (a) The generating function is $\Sigma_k a_k x^k = -1/(1-x)$.

(b) Here we have : $1/(1-2x) - 1 = 2x/(1-2x)$.

(c) The generating function of this is clearly (refer to page 343 : because $(n-1) = (n+1) - 2$), $1/(1-x)^2 - 2/(1-x)$.

(d) If the generating function of this be $f(x)$ then, $xf(x) = e^x - 1$, (from the expression for e^x). Hence, $f(x) = (e^x - 1)/x$.

(e) Let $f(x)$ be the generating function we seek. From Table.1 we know that $1/(1-x)^3 = \Sigma_{n=0}^{\infty} {}^{n+2}C_2 x^n$, and that is almost what we have here. To transform this to $f(x)$, we need to factor out x^2, and change the variable of summation :

$$\frac{1}{(1-x)^3} = \Sigma_{n=0}^{\infty} {}^{n+2}C_2 x^n = \frac{1}{x^2}\Sigma_{n=0}^{\infty} {}^{n+2}C_2 x^{n+2} = \frac{1}{x^2}\Sigma_{n=2}^{\infty} {}^{n}C_2 x^n = \frac{1}{x^2}.(f(x) - f(0) - f(1))$$

Noting that $f(0) = f(1) = 0$ by definition, we have $f(x) = x^2/(1-x)^3$.

(f) The generating function for this is clearly binomial. So, we use Table.1,

$$\Sigma_{n=0}^{\infty}{}^{10}C_{n+1}x^n = \Sigma_{n=1}^{\infty}{}^{10}C_n x^{n-1} = \frac{1}{x}\Sigma_{n=1}^{\infty}{}^{10}C_n x^n = \frac{1}{x}((1+x)^{10}-1)$$

Q#32 (pg.352) First let $G(x) = \Sigma_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \Sigma_{k=0}^{\infty} a_k x^{k+1} = \Sigma_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$). Thus, $G(x) - 7xG(x) = \Sigma_{k=0}^{\infty} a_k x^k - \Sigma_{k=1}^{\infty} 7a_{k-1}x^k = a_0 + \Sigma_{k=0}^{\infty}(a_k - 7a_{k-1})x^k = a_0 + 5$ because of the given recurrence relation and initial condition. Thus, $G(x)(1-7x) = 5$, so $G(x) = 5/(1-7x)$. Then from Table. 1 on page 343 we know that $a_k = 5.7^k$.

Q#36 (pg. 352) Let $G(x) = \Sigma_{k=0}^{\infty} a_k x^k$. Then, $xG(x) = \Sigma_{k=0}^{\infty} a_k x^{k+1} = \Sigma_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and $x^2G(x) = \Sigma_{k=0}^{\infty} a_k x^{k+2} = \Sigma_{k=2}^{\infty} a_{k-2}x^k$. Thus,

$$G(x) - xG(x) - 2x^2G(x) = \Sigma_{k=0}^{\infty} a_k x^k - \Sigma_{k=1}^{\infty} a_{k-1}x^k - \Sigma_{k=2}^{\infty} 2a_{k-2}x^k =$$
$$a_0 + a_1x - a_0x + \Sigma_{k=2}^{\infty} 2^k x^k = 4 + 8x + \frac{1}{(1-2x)} - 1 - 2x = \frac{4-12x^2}{1-2x},$$

because of the given recurrence relation and the initial conditions, and Table 1. on page 343. Since the left hand side of this equation factors as $G(x)/(1-2x)(1+x)$ we have, that, $G(x) = (4-12x^2)/((1+x)(1-2x)^2)$. At this point, we have to use partial fractions to break up the denominator. Setting

$$\frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2},$$

multiplying through by the common denominator, and equating coefficients, we find that $A = -8/9, B = 38/9, C = 2/3$. Thus

$$G(x) = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2} = \Sigma_{k=0}^{\infty}\left(-\frac{8}{9}.(-1)^k + \frac{38}{9}.2^k + \frac{2}{3}(k+1).2^k\right)x^k$$

by application of the relevant from Table. 1, pg.343. Therefore,

$$a_k = (-8/9)(-1)^k + (38/9)2^k + (2/3)(k+1)2^k.$$

Thus, putting $k = 2$ we have that $a_2 = 24$.