Homework 3 Solutions.

$Q\#6$. (pg. 359) (a). Since $A_1 \subseteq A_2 \subseteq A_3$, $A_1 \cup A_2 \cup A_3 = A_3$. Hence, the size of the union is 10000.
(b). If the sets are pairwise disjoint, then the size of the union is the sum of the sizes, that is, 11100.
(c). Now, we need to apply the principle of inclusion-exclusion. We are given that the pairwise intersections contain 2 elements, and the threewise intersection just containts 1 element. Hence, by the principle of inclusion-exclusion for three sets, we have that the size of the union is equal to $100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11095$.

$Q\#12$. (pg. 360) If a number has to be both a square and a cube, then the only possibility is that it is a 6^{th} power. Now, such numbers not exceeding 1000 are, 1, $2^6 = 64$, $3^6 = 729$ (4^6 exceeds 1000). Hence there are just 3 such positive integers, which are both squares as well as cubes. Now, there are $\sqrt{1000} = 31$ squares) and $\sqrt[3]{1000} = 10$ cubes. Therefore, the answer is $31 + 10 - 3 = 38$ (by inclusion exclusion).

$Q\#18$. (pg. 360) The number of terms is equal to $2^{10} - 1 = 1023$.

$Q\#4$. (pg 367) Define sets of solutions

- $B_1 = \{x_1 \leq 3\}$
- $B_2 = \{x_2 \leq 4\}$
- $B_3 = \{x_3 \leq 5\}$
- $B_4 = \{x_4 \leq 8\}$

one set corresponding to each variable. Clearly the set of solutions we are looking for happens to be the intersection of all these 4 sets. So, this is a case for the principle of inclusion exclusion, but we also know that the principle can be more suitably applied to find out the size of the union of sets. So, first we apply DeMorgan's to go over to the complements, and we need to find out $|B_i^c|$, the pairwise intersection sizes and so on. For instance $|B_1^c| = {}^{16}C_{13}$. Performing all the calculations, and then writing out the principle of inclusion exclusion, the answer comes out to be 20.

$${}^{(4+17-1)}C_{17} - {}^{(4+13-1)}C_{13} - {}^{(4+12-1)}C_{12} - {}^{(4+11-1)}C_{11} - {}^{(4+8-1)}C_8 + {}^{(4+8-1)}C_8 + {}^{(4+7-1)}C_7 + {}^{(4+4-1)}C_4 + {}^{(4+6-1)}C_6 + {}^{(4+3-1)}C_3 + {}^{(4+2-1)}C_2 - {}^{(4+2-1)}C_2 = 20$$

$Q\#6$. (pg. 367) We try to first count the integers which are not squarefree. For this, we define the sets

- $N_2 = \{n : n < 100 \text{ and } 2^2 \text{ divides } n\}$

- $N_3 = \{n : n < 100 \text{ and } 3^2 \text{ divides } n\}$
- $N_5 = \{n : n < 100 \text{ and } 5^2 \text{ divides } n\}$
- $N_7 = \{n : n < 100 \text{ and } 7^2 \text{ divides } n\}$

Observe, we don't need to define more sets, because if you have a number which is less than 100 and not squarefree, then the number whose square divides it must be less than 10. Clearly, the set of numbers which are less than 100 and not squarefree, is simply the union of the above 4 sets. So all we need to do is to count the number of elements in the union of the sets above. Now, once having defined N_2 say, we dont need to define N_4 any more, cause $N_4 \subset N_2$ (so N_4 doesn't add anything new to the union, nothing that N_2 doesn't already contribute). Now, taking ceilings, we have that $|N_2| = 24$, $|N_3| = 11$, $|N_5| = 3$, $|N_7| = 2$. Now for the pairwise intersections, we have that $|N_5 \cap N_7| = 0$, in fact all the pairwise intersections are empty except for $|N_2 \cap N_3| = 2$. Hence all the threewise or other intersections are empty too. Now, applying the principle of inclusion exclusion, we have that the size of the union is equal to $24+11+3+2-2 = 38$. Hence the number of numbers less than 100 which aren't squarefree is just 38. Hence, the number which are squarefree is $99 - 38 - 1 = 60$ (discounting 1 as a squarefree number).

Q#26. (pg. 368) Since we are talking derangements here, no element is supposed to fall in its own place (natural place). If the derangement ends with the integers 1, 2 and 3 in some order, that means that the first half of the permutation has $4, 5, 6$ in some order, and the latter half $1, 2, 3$ in some order. So, we can permute the first half of the derangement that we have, and the latter half, independently, to generate all the derangements we want to count. So thats just $3! \times 3! = 36$ derangements.