1 Integer equations and inequalities with constraints

Consider the following linear equation in $n$ variables:

$$x_1 + \ldots + x_n = k,$$

where $k$ is a nonnegative integer. This time we are interested in finding the number of nonnegative integer solutions to this equation, i.e. values of $x_1, \ldots, x_n$ that satisfy the equation with the additional constraint that

$$x_1 \geq r_1,$$
$$x_2 \geq r_2,$$
$$\vdots$$
$$x_n \geq r_n,$$

where $r_1, \ldots, r_n$ are non-negative integers. Obviously, if $r_1 + r_2 + \ldots + r_n > k$ then there are no possible solutions (do you see why?). So might as well assume the more interesting case when $\sum r_i \leq k$.

We can “construct” a solution to the above set of equations and constraints in two steps:

- First we “give” $r_i$ out of $k$ “units” to every variable $x_i$ to fulfil its minimum requirement, i.e. to satisfy $x_i \geq r_i$. After this, we will be left with $k - (r_1 + r_2 + \ldots + r_k)$ units.

- Now, note that the remaining $k - \sum r_i$ units can be distributed in any way we want between $x_1, \ldots, x_n$. For each $i$, define

$$x'_i = x_i - r_i.$$

$x'_i$ can be thought of as the “surplus” that variable $x_i$ receives over and above its minimum requirement of $r_i$. How much should all the surpluses add up to? It is easy to see that

$$x'_1 + x'_2 + \ldots + x'_n = k - (r_1 + r_2 + \ldots r_n),$$

i.e. the surpluses add up to the amount of “units” remaining after the minimum needs are fulfilled in step 1. Also, note that $x'_i \geq 0$ for each $i$ (surplus cannot be negative). Thus, to finish, we now assign values to the surplus variables $x'_i$ so that they obey the equation above.
Once we have assigned values to the surplus variables, we can find values for the original variables using the relationship

\[ x_i = x'_i + r_i \]

for each \( i \)

The above steps show us that given a solution to

\[ x'_1 + \ldots + x'_n = k - (r_1 + \ldots + r_n) \]

under the constraints that \( x'_i \geq 0 \) for each \( i \), we can use that solution to obtain a solution for

\[ x_1 + \ldots + x_n = k \]

under the constraints \( x_i \geq r_i \). Conversely, it is not hard to see that given any valid solution to

\[ x_1 + \ldots + x_n = k \]

under the constraints \( x_i \geq r_i \) for each \( i \), we can create a solution for

\[ x'_1 + \ldots + x'_n = k - \sum_i r_i \]

under the constraints \( x'_i \geq 0 \) for each \( i \), by simply setting

\[ x'_i = x_i - r_i. \]

If \( A \) represents the set of all solutions to \( x_1 + \ldots + x_n = k \) under the constraint \( x_i \geq r_i \), and \( B \) represents the set of all solutions to \( x'_1 + \ldots + x'_n = k - \sum_i r_i \) under the constraint that \( x'_i \geq 0 \), then what the above discussion shows is that there is a bijection between \( A \) and \( B \), and so using the bijection method it follows that \( |A| = |B| \).

We know from previous lectures that the number of solutions to

\[ x'_1 + \ldots + x'_n = k - \sum_i r_i \]

under the constraints \( x'_i \geq 0 \), is

\[ |B| = \binom{k - (r_1 + \ldots + r_n) + n - 1}{n - 1}. \]

We can conclude that the number of solutions to

\[ x_1 + \ldots x_n = k \]

with the constraints \( x_i \geq r_i \) is also the same.

**Theorem 1.** The number of solutions to

\[ x_1 + \ldots + x_n = k \]

under the constraint that \( x_i \geq r_i \) for each \( i \), where \( r_i \geq 0 \) is an integer, and \( \sum i r_i \leq k \), is

\[ \binom{k - \sum_i r_i + n - 1}{n - 1}. \]
If we were dealing with the inequality
\[ x_1 + \ldots + x_n \leq k \]
under the constraint that \( x_i \geq r_i \) where \( r_i \geq 0 \) are integers such that \( \sum_i r_i \leq k \), then using a similar analysis (i.e., using surplus variables \( x'_i = x_i - r_i \)) as above we can conclude that

**Theorem 2.** The number of solutions to
\[ x_1 + \ldots + x_n \leq k \]
under the constraint that \( x_i \geq r_i \) for each \( i \), where \( r_i \geq 0 \) is an integer, and \( \sum_i r_i \leq k \), is
\[ \left( k - \sum_i r_i + n \right) \]

**Question.** Find the number of integer solutions to the inequality
\[ x_1 + x_2 + x_3 + x_4 \leq 20, \]
under the constraint that \( x_1, x_3, x_4 \geq 1 \), and \( 0 \leq x_2 \leq 4 \).

**Proof.** Let \( A \) be the set of all solutions to the given inequality and constraints. Let \( U \) be the set of all solutions to
\[ x_1 + x_2 + x_3 + x_4 \leq 20 \]
under the constraints \( x_1, x_3, x_4 \geq 1 \) and \( x_2 \geq 0 \) (note the difference). Notice that any solution in \( A \) is also a solution in \( U \), because the first inequality has the constraint \( 0 \leq x_2 \leq 4 \) which also trivially satisfies the constraint \( x_2 \geq 0 \) for the second inequality (all the other constraints are identical for both inequalities). Thus, \( A \subseteq U \). Let us now use the difference method
\[ |A| = |U| - |U - A|. \]
Let us try to understand what the set \( U - A \) is. This set contains solutions to
\[ x_1 + x_2 + x_3 + x_4 \leq 20 \]
where \( x_1, x_3, x_4 \geq 0 \) and \( x_2 \geq 0 \) is satisfied, but the constraint \( x_2 \leq 4 \) is violated. This is because had the constraint \( x_2 \leq 4 \) also been satisfied then this solution would also be a solution in \( A \) but that’s impossible since are looking at solutions in \( U - A \) which by definition cannot contain solutions from \( A \). Thus, \( U - A \) is the set of all solutions to
\[ x_1 + x_2 + x_3 + x_4 \leq 20 \]
under the constraints \( x_1, x_3, x_4 \geq 1 \), and \( x_2 > 4 \) or \( x_2 \geq 5 \) (we are looking at integer solutions). Define
\[ x'_1 = x_1 - 1 \]
\[ x'_3 = x_3 - 1 \]
\[ x'_4 = x_4 - 1 \]
\[ x'_2 = x_2 - 5. \]
Note that these are exactly the surplus variables we had defined earlier in our discussion. Recall that the surplus variables can be any integer greater than or equal to 0, i.e. \( x'_i \geq 0 \) for all \( i \): surplus can be zero, since it is whatever a variable gets after its minimum requirement has been fulfilled. Also, we can see that the surpluses must add up to at most whatever is left after giving out the minimum needs\(^1\). In this case, the remaining value is

\[
20 - 1 - 5 - 1 - 1 = 12.
\]

Thus, the surplusses must satisfy

\[
x'_1 + x'_2 + x'_3 + x'_4 \leq 12,
\]

under the constraint that \( x'_1, x'_2, x'_3, x'_4 \geq 0 \). Based on our discussions, earlier any solution to this inequality gives us a solution in \( U - A \), and vice-versa. Thus, all we need to do is to find the number of solutions to the inequality involving surplus variables. We know that from the star and bars method that the number of solutions to

\[
x'_1 + x'_2 + x'_3 + x'_4 \leq 12
\]

under the constraint that \( x'_i \geq 0 \) for all \( i \), is

\[
\binom{12 + 4}{4} = \binom{16}{4},
\]

and so

\[
|U - A| = \binom{16}{4}.
\]

Let us now find \( |U| \). \( U \) is all solutions to

\[
x_1 + x_2 + x_3 + x_4 \leq 20
\]

under the constraints that \( x_1, x_3, x_4 \geq 1 \) and \( x_2 \geq 0 \). Again, either using the formula given in Theorem 2 or by using the same analysis as above, we can show that

\[
|U| = \binom{17 + 4}{4} = \binom{21}{4}.
\]

Then, using the difference method

\[
|A| = |U| - |U - A| = \binom{21}{4} - \binom{16}{4}.
\]

\( \square \)

**Note:** In the above proof, you could have also directly used the formula in Theorem 2 to compute \( |U - A| \) but I did not do that just to illustrate the method for inequalities.

\(^1\)They don’t need to be equal to the remaining units since we are finding solutions to an inequality and so we may not distribute everything between the surplus variables and might choose to “discard” some units.
Here is another problem (this one is from your book; and we had discussed this in the first problem set) that can seem totally unrelated to any of the applications of the stars and bars method we have seen so far:

**Question**. Suppose there are 20 books arranged in a row on a book shelf. In how many ways can you choose 6 books so that no two of the chosen books are adjacent to each other on the rack?

At first glance, this problem might not seem to have anything to do with what we have been discussing so far. A little bit of rephrasing makes this illusion disappear.

Note that the most obvious way of specifying a selection/choice of 6 books is to literally specify the “index” of the book if one were to start counting from 1 starting at the left most book on the shelf.

A more non-obvious way would be the following: specify how many unchosen books are there are to the left of the first chosen book, then specify how many unchosen books there are between the first and second chosen book, then the number of unchosen books between the second and third chosen book, ..., and finally the number of unchosen books to the right of the last (sixth) chosen book. So, I can set this up in the following way: let \( x_1 \) denote the number of books to the left of the first chosen book, \( x_2 \) denote the number of books between the first and second chosen books, and so on, with \( x_7 \) representing the number of books to the right of the sixth chosen book.

What conditions do we want to impose on these variables? Well, first of all there must be 14 unchosen books in all, and each of those 14 books must be either between two chosen books, or to the left of the first chosen book or to the right of the last chosen (sixth) book, and thus: \( x_1 + x_2 + \ldots + x_7 = 14 \).

Furthermore, we want that no two books be adjacent, so there must be at least one unchosen book between the first and second chosen book, so we want \( x_2 \geq 1 \), and similarly we want \( x_3, \ldots, x_6 \geq 1 \).

How about \( x_1 \) and \( x_7 \)? There is no constraint on them: there could be as low as zero (it is possible to choose the first book as the leftmost book on the shelf, or the last book as the rightmost book on the shelf), and so \( x_1, x_7 \geq 0 \).

Thus, choosing 6 non-adjacent books from the 20 books on the shelf is equivalent to find a solution to

\[
x_1 + \ldots + x_7 = 14
\]

under the constraints \( x_2, \ldots, x_6 \geq 1 \) and \( x_1, x_7 \geq 0 \). Using Theorem 1, the number of solutions to this equation, and hence the number of ways of picking the books, is

\[
\binom{14 - 5 + 7 - 1}{7 - 1} = \binom{15}{6}.
\]