1 Identical rooks on the chessboard

In the previous lecture you saw in how many ways we can place a black and a white rook on an $8 \times 8$ chess board so that the rooks cannot attack each other. What happens if the two rooks are identical (indistinguishable)?

Let us formalize the approach we took in the case when we had two rooks of different colors. First, we can assume that the columns of the board are labeled 1 to 8 from left to right, and the rows are labeled 1 to 8 from top to bottom, so every position on the board can be described using coordinates $(r, c)$. Then to “encode” an arrangement of two rooks on the board we can basically specify the position of the black rook and the position of the white rook. To do this formally, we can use a tuple $((r_1, c_1), (r_2, c_2))$ such that the first position in the 2-tuple, i.e. $(r_1, c_1)$, is the position of the black rook, and the second position in the 2-tuple, i.e. $(r_2, c_2)$, is the position of the white rook. A 2-tuple of positions turns out to be the best “data structure” to describe an arrangement in this case (why?), and the problem boils down to counting the number of possible tuples $((r_1, c_1), (r_2, c_2))$ such that a rook in position $(r_1, c_1)$ cannot attack a rook in the position $(r_2, c_2)$, which leads to the answer $64 \times 49$.

Now suppose the rooks are of the same color and hence indistinguishable. Should we still use a 2-tuple containing two positions to describe an arrangement of the two rooks on the chessboard? The problem with using 2-tuples is that they create some redundancy: consider an arrangement where one of the rooks is at position $(1,1)$ and the other is at $(7,2)$ (and so they are in non-attacking positions); both $((1,1), (7,2))$ and $((7,2), (1,1))$ describe the same arrangement (why?)! Whereas in the case of two rooks of distinct colors, the above 2-tuples described different arrangements! This means if we simply reported the same answer (i.e., $64 \times 49$) for the case of identical rooks we would be double-counting the number of arrangements (why?).

Is there a better\footnote{By better we mean a data structure that eliminates any redundancy.} “data structure” to describe an arrangement of identical rooks on the board? The answer is to use a set containing the two positions of the rooks instead of using a 2-tuple containing the positions (why?). Thus, the number of ways to place two identical rooks on the chessboard so that they don’t attach each other is equal to the number of sets $\{(r_1, c_1), (r_2, c_2)\}$ such that the rook in position $(r_1, c_1)$ does not attack the rook in position $(r_2, c_2)$. Unfortunately, counting the number of such sets is a little hard and we shall come back to this problem after introducing some more techniques.
2 Division rule/Generalized bijection method

Definition 1. A function $f: A \rightarrow B$ is called a $k$-to-1 function if for every $b \in B$ there exist $k$ distinct elements $a_1, \ldots, a_k$ such that all of them are mapped to $b$ by $f$, i.e. $f(a_1) = f(a_2) = \ldots = f(a_k) = b$.

If a function is $k$-to-1 with $k = 1$, then it is a bijection!
Consider the function $f: \mathbb{Z} \setminus 0 \rightarrow \mathbb{N} \setminus \{0\}$, i.e. the domain is all integers except 0 and the codomain is all natural numbers except 0, such that $f(x) = |x|$ where $|x|$ denotes the absolute value of $x$. Then $f$ is a 2-to-1 function: for every non-zero natural number $n$, both $n$ and $-n$ are mapped to $n$ by $f$!

Lemma 2. If $A$ and $B$ are finite sets, and $f: A \rightarrow B$ is a $k$-to-1 map, then $|A| = k \cdot |B|$.

Proof. Let us assume that $|B| = m$ and $B = \{b_1, \ldots, b_m\}$. Note that by the definition of a function, every element in $A$ must be mapped to exactly one element in $B$. This means we can partition the elements of $A$ into $m$ parts $A_1, A_2, \ldots, A_m$ where the part $A_i$ is the set of all elements in $A$ that get mapped to $b_i$ by $f$. By the partition rule, we have that $|A| = \sum_{i=1}^{m} |A_i|$. What are the sizes of the $A_i$s? We know that exactly $k$ elements get mapped to every element in $B$, and this means that for every $i$, $|A_i| = k$. This means that $|A| = k \cdot m = k \cdot |B|$. This finishes the proof. \qed

This suggests a new method for finding the size of set $B$ that might be hard to count directly. Find an easy to count set $A$ such that there is a $k$-to-1 function from $A$ to $B$ for some positive integer $k$, and then using the above lemma, we have that $|B| = |A|/k$. This method is called the division rule or the generalized bijection method.

Question. Suppose five knights are to be seated around a round table. How many distinct ways of seating them are there? Two arrangements are considered to be the same if every knight has the same left-neighbor and right-neighbor in both the arrangements.

Proof. Let $B$ be the set of ways of seating the 5 knights around the table. Let $A$ be the number of ways of making the 5 knights stand in a line, one behind the other. We want to find $|B|$, but instead we will find $|A|$, set up a $k$-to-1 function between $A$ and $B$, and then use the division rule to find $|B|$.

First note that $|A| = 5!$ since it’s just the number of ways of permuting 5 distinct objects. There is a natural way of converting an arrangement of the 5 knights in a line to a seating plan around the table: if the knights are standing in the order $(k_1, k_2, k_3, k_4, k_5)$, then we seat them clockwise around the table in the order $k_1, k_2, k_3, k_4, k_5$ such that $k_1$ and $k_3$ are now neighbors around the table (in particular, $k_1$ has $k_2$ to their left and $k_5$ to their right, $k_2$ has $k_1$ to their right and $k_3$ to their left, and so on). Let’s call this mapping from permutations to seating arrangements $f: A \rightarrow B$.

Notice that all of the following permutations will get mapped to exactly the same seating arrangement: $(k_1, k_2, k_3, k_4, k_5), (k_2, k_3, k_4, k_5, k_1), (k_3, k_4, k_5, k_1, k_2), (k_4, k_5, k_1, k_2, k_3)$, and $(k_5, k_1, k_2, k_3, k_4)$. In general, you can convince yourself that for every seating arrangement in $B$, there will be exactly
5 permutations in \( A \) that will map into it. This means that \( f \) is a 5-to-1 map and thus, using the division rule, we have that

\[
|B| = \frac{|A|}{k} = 4!.
\]

2.1 Returning to the identical rooks problem

Let \( A \) be the set of all 2-tuples of positions \(((r_1, c_1), (r_2, c_2))\) such that a rook in the first position, i.e. \((r_1, c_1)\), will not attack a rook in the second position, i.e. \((r_2, c_2)\). On the other hand, let \( B \) be the set containing all sets of two positions \{\((r_1, c_1), (r_2, c_2)\)\} such that a rook in position \((r_1, c_1)\), will not attack a rook in position \((r_2, c_2)\).

Based on our previous discussion, \(|A|\) is the basically the number of ways to place a white rook and a black rook on the board so that they don’t attack each other (i.e., the case of distinct rooks), and \(|B|\) is the number of ways of placing identical rooks on the board so that they don’t attack each other (i.e., the case of indistinguishable rooks). Instead of trying to compute \(|B|\), we will define a \( k \)-to-1 function \( f : A \to B \), and then, since we already know \(|A|\), we can compute \(|B|\) using the division rule, i.e. \(|B| = \frac{|A|}{k}\).

We will define \( f : A \to B \) simply as the function that takes a 2-tuple \(((r_1, c_1), (r_2, c_2))\) in \( A \) and transforms into the set \{\((r_1, c_2), (r_2, c_2)\)\}. Note that for every set \{\((r_1, c_2), (r_2, c_2)\)\} in the codomain \( B \), there are exactly two tuples in \( A \) that are mapped to it be \( f \), namely

\[
\begin{align*}
  f(((r_1, c_1), (r_2, c_2))) &= \{(r_1, c_2), (r_2, c_2)\} \\
  f(((r_2, c_2), (r_1, c_1))) &= \{(r_1, c_2), (r_2, c_2)\}
\end{align*}
\]

and so \( f \) is a 2-to-1 function. This means that

\[
|B| = \frac{|A|}{2} = \frac{64 \times 49}{2}.
\]

3 Permutations with repetitions

**Question.** How many distinct permutations of the word SYSTEMS are there?

**Proof.** Notice that the answer isn’t simply 7! (Why?). The number of ways of permuting \( n \) distinct objects is \( n! \), but we have to be more careful when we have repetitions. In this case, we have a repetition: there are three copies of \( S \). We can use the division rule to solve this problem. Let \( B \) be the set of distinct permutations of \( SYSTEMS \). Let \( A \) be the set of permutations of the string \( S_1YS_2TEMS_3 \) (here the three Ss are distinguishable from each other). We know that \(|A| = 7!\) because we are simply arranging 7 distinct objects in all possible ways.

Let us now give a map \( f : A \to B \) that is \( k \)-to-1 for some value of \( k \). We can then find \(|B|\) using the division rule (so \( f \) maps a permutation of \( S_1YS_2TEMS_3 \) to an permutation of the letters of \( SYSTEMS \)). Here is how \( f \) is defined: given a permutation of \( S_1YS_2TEMS_3 \), \( f \) simply drops the subscripts of the three Ss in the permutation to get a permutation of \( SYSTEMS \), e.g., \( S_1TMS_3EYS_2 \) is mapped to \( STMSEYS \) by \( f \). It’s easy to see that exactly 3! different
permutations of $S_1YS_2TEMS_3$ in $A$ will map to any given permutation of $SYSTEMS$ in $B$ — the order in which $S_1, S_2, S_3$ occur in a permutation does not make a difference! For example, all six of the following will map to $STMSEYS$:

- $S_1TMS_3EYS_2$
- $S_1TMS_2EYS_3$
- $S_2TMS_3EYS_1$
- $S_2TMS_1EYS_3$
- $S_3TMS_1EYS_2$
- $S_3TMS_2EYS_1$

Thus, $f$ is a $k$-to-1 map with $k = 3!$, and so

$$|B| = \frac{|A|}{k} = \frac{7!}{3!}.$$