Let us recall a property of binomial coefficients that we studied in one of the earlier lectures.

**Theorem 1.** For all integers $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

**Proof.** One way to prove this is by using the formula for binomial coefficients (I leave it as an exercise). Another way to prove it is the following. Let $S_k$ be the set of all binary strings of length $n$ with exactly $k$ ones, and let $S_{n-k}$ be the set of all binary strings with exactly $n-k$ ones. We will define a bijection $f : S_k \rightarrow S_{n-k}$. $f$ is simply defined as the function that takes a binary string of length $n$ with $k$ ones and flips all its $n$ bits. For example, if $n = 4$ and $k = 3$, then $f(1011) = 0100$.

Clearly, a string of length $n$ with $k$ ones will get mapped by $f$ to a string with $n-k$ ones. You can convince yourself that this is indeed a bijection (i.e., no two strings in $S_k$ are mapped by $f$ to the same string in $S_{n-k}$, and every string in $S_{n-k}$ has at least one string mapped to it). □

## 1 Binomial theorem

Recall that $(x+y)^2 = x^2 + y^2 + 2xy$. Actually, there is a lot going on when one tries to derive that:

- $(x+y)^2$ is nothing but $(x+y) \times (x+y)$. Since multiplication is **distributive** over addition, there will be four terms coming from the four possible multiplications, and then all we be added up.

- Basically, each of the four terms will consist of exactly one entry each from the two copies of $(x+y)$. For the first copy, we have the choice of picking up either $x$ or $y$, and we have the same choices for the second copy. This gives $2 \times 2 = 4$ terms in total.

- The four terms are $xx + xy + yx + yy$. Now recall that $xx = x^2$ and $yy = y^2$. Also, because of multiplication being **commutative** we have that $xy = yx$, and so we get $x^2 + y^2 + 2xy$.

One can now do the same for higher powers of $(x+y)$. How about $(x+y)^n$? This is nothing but $n$ copies of $(x+y)$ multiplied together. Again, using distributivity of multiplication over addition, we will get $2^n$ terms: there are $n$ copies of $(x+y)$, and we have two choices for the first copy, two for the second copy, and so on.

Obviously, as before, many terms will “collapse” to the same expression because of commutativity ($xy = yx$, and then combining consecutive $x$s and $y$s into powers). This means that, in the final
simplified expression for \((x + y)^n\), different terms will have different coefficients in front of them depending on how many of the \(2^n\) original terms collapse to them.

Let’s ask the question, what’s the coefficient of \(x^k y^{n-k}\)? In other words, how many of the \(2^n\) initial terms collapse to \(x^k y^{n-k}\). Clearly, only those terms that have exactly \(k\) \(x\)s in them will collapse to \(x^k y^{n-k}\), so how many terms have exactly \(k\) \(x\)s?

Recall that we had to pick between \(x\) and \(y\) in each of the \(n\) copies of \((x + y)\). Thus, the terms that give us \(x^k y^{n-k}\) must correspond to outcomes of the picking process where we decided to pick \(x\) in exactly \(k\) of the copies of \((x + y)\), and picked \(y\) in the rest. How many such outcomes are there? It’s exactly the same as deciding in which \(k\) copies of \((x + y)\) (out of the \(n\) possible) will we pick \(x\), and there are \(\binom{n}{k}\) ways of doing that. So the coefficient of \(x^k y^{n-k}\) must be \(\binom{n}{k}\). Since there was nothing special about \(n\) and \(k\) in the above argument, we can conclude:

**Theorem 2** (Binomial theorem). Let \(n \geq 0\) be an integer, then we have that

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

## 2 Sum of all binomial coefficients

The binomial theorem lets us derive some interesting things about binomial coefficients. Let us first ask the following question what is the sum of all binomial coefficients for a fixed \(n\), i.e.

\[
\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}.
\]

We can answer this problem in two ways. The first one is combinatorial: recall that \(\binom{n}{k}\) just denoted the number of size \(k\) subsets of the set \(\{1, \ldots, n\}\). So, it seems like we are adding the number of subsets of size 0, the number of subsets of size 1, and so on up till the number of subsets of size \(n\). But if you think about it this accounts for all possible subsets of \(\{1, \ldots, n\}\), and thus the count must add up to the total number of different possible subsets of \(\{1, \ldots, n\}\), which we know is \(2^n\).

A more algebraic way of seeing this is take the statement of the binomial theorem and set \(x = y = 1\). Clearly, the LHS of the binomial theorem (as in Theorem 2) becomes \(2^n\), while the RHS becomes the sum of all the binomial coefficients, and thus we can conclude the following:

**Theorem 3.**

\[
2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}
\]

## 3 Sum of even binomial coefficients equals the sum of odd binomial coefficients

We say that a binomial coefficient \(\binom{n}{k}\) is **even** if \(k\) is even, otherwise we say that the binomial coefficient is **odd**. We will now try to understand what happens if we sum up all the even binomial coefficients, i.e. compute the value of

\[
\sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k}.
\]
What happens if we set \( y = 1 \) and \( x = -1 \) in Theorem 2? The LHS becomes zero, and we get

\[
0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.
\]

Now notice that \((-1)^k\) becomes 1 whenever \( k \) is even, and becomes \(-1\) whenever \( k \) is odd. We can collect all the positive terms together and all the negative terms together and we can write:

\[
0 = \sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} - \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k},
\]

and so we can conclude the following

**Theorem 4.**

\[
\sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} = \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k}.
\]

Note that Theorem 2 can be rewritten as follows by grouping all the even binomial coefficients and all the odd binomial coefficients:

\[
2^n = \sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} + \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k}.
\]

We can combine the above equation with Theorem 4 to conclude the following:

**Theorem 5.**

\[
\sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} = \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k} = 2^{n-1}.
\]

There is another way of interpreting this statement:

\[
\sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k}
\]

is nothing but the number of even size subsets of \( \{1, \ldots, n\} \) (do you see why?) and

\[
\sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k}
\]

is the number of odd size subsets of \( \{1, \ldots, n\} \), and the above expression tells us that the number of odd size subsets is equal to the number of even size subsets, and both are equal to \(2^{n-1}\) (exactly half of the total number of subsets of \( \{1, \ldots, n\} \)). This can also be shown using the bijection method but you will be doing that as part of the problem set for this week.