1 Middle binomial coefficients

For any integer \( n \geq 0 \), the number of binomial coefficients of the form \( \binom{n}{k} \) is \( n+1 \): \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n} \). When \( n \) is even, the “middle” binomial coefficient is \( \binom{n}{n/2} \). In this case, the middle binomial coefficient has exactly \( \frac{n}{2} \) binomial coefficients before it, i.e. \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n/2-1} \), and exactly \( \frac{n}{2} \) binomial coefficients after it, i.e. \( \binom{n}{n/2+1}, \ldots, \binom{n}{n} \).

When \( n \) is odd, the story is a little different, there will be two binomial coefficients in the middle: \( \binom{n}{n/2} \) and \( \binom{n}{n/2+1} \), and they will have exactly \( \frac{n-1}{2} \) binomial coefficients before them, i.e. \( \binom{n}{0}, \ldots, \binom{n}{n/2-1} \), and will have exactly \( \frac{n-1}{2} \) binomial coefficients after them, i.e. \( \binom{n}{n/2+2}, \ldots, \binom{n}{n} \).

2 The behavior of binomial coefficients as \( k \) varies

We will now try to understand how \( \binom{n}{k} \) changes as we vary \( k \) from 0 to \( n \). Let us define the function \( \text{Binom}_n : \{0, \ldots, n\} \to \mathbb{N} \) as \( \text{Binom}_n(k) = \binom{n}{k} \).

If we use the formula

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!},
\]

we can plot the function \( \text{Binom}_5 \) as the value of \( k \) varies from 0 to 5:

You can see from the above plot that the two middle coefficients \( (k = 2 \text{ and } k = 3) \) are the largest in value, and that the value of \( \binom{5}{k} \) increases as \( k \) varies from 0 to 2, then doesn’t change when we
move from 2 to 3, and then drops as we move from 3 to 5. It should not be surprising that the plot has some nice symmetry to it. In particular, we can see that \( \binom{5}{0} = \binom{5}{5}, \binom{5}{1} = \binom{5}{4}, \) and \( \binom{5}{2} = \binom{5}{3}. \) We have seen this before; we know that for all \( n \) and \( k \)

\[
\binom{n}{k} = \binom{n}{n-k}.
\]

Thus, because of this property, it will always be the case that the plot of \( \text{Binom}_n(k) \) will be symmetric about the middle binomial coefficient(s) (i.e., about the \( k = \frac{n}{2} \) vertical line in the plot).

We will now plot \( \text{Binom}_{10} \):

![Binomial Coefficient Graph](image.png)

Note that since \( n = 10 \) is even, we have only one middle binomial coefficient \( (k = 5) \) instead of two (as in the odd case of \( n = 5 \)), and it is also the largest in value. The behavior is similar to the case of \( n = 5 \): as \( k \) goes from 0 to \( \frac{n}{2}, \) \( \binom{n}{k} \) increases, but then it drops as we go from \( \frac{n}{2} \) to \( n. \) Again, one can observe the symmetry of the binomial coefficients about \( k = \frac{n}{2}. \)

We will now move on to formally proving some facts about the behavior of binomial coefficients \( \binom{n}{k} \) as \( k \) increases. We will first consider the case of odd \( n. \)

**Theorem 1.** Let \( n \) be an odd positive integer. Then if \( 0 \leq k < \frac{n-1}{2}, \)

\[
\binom{n}{k+1} > \binom{n}{k},
\]

if \( k = \frac{n-1}{2} \) then \( \binom{n}{k+1} = \binom{n}{k}, \) and when \( \frac{n-1}{2} < k < n, \)

\[
\binom{n}{k+1} < \binom{n}{k}.
\]

**Proof.** For \( 0 \leq k \leq n-1, \) define \( r_k \) as follows:

\[
r_k = \frac{\binom{n}{k+1}}{\binom{n}{k}},
\]

i.e., the ratio of the \( (k+1)^{th} \) binomial coefficient to the \( k^{th} \) binomial coefficient. We can simplify \( r_k \) as

\[
r_k = \frac{n}{k+1} \frac{k!(n-k)!}{(n-k)(n-k-1)!} = \frac{n - k}{k + 1}.
\]
(Can you explain why the last equality in the above series of equalities is true?) Note that whenever \( r_k > 1 \), then the \((k+1)\text{th}\) binomial coefficient is larger than the \(k\text{th}\) one, and it’s the other way around when \( r_k < 1 \), so we just need to understand the behavior of \( r_k \) with \( k \):

\[
r_k = \frac{n-k}{k+1} < 1 \iff k > \frac{n-1}{2},
\]

and

\[
r_k = \frac{n-k}{k+1} > 1 \iff k < \frac{n-1}{2}.
\]

Also, it’s easy to see that the \((k+1)\text{th}\) and \(k\text{th}\) binomial coefficients are equal when \( r_k = 1 \) which can only happen when \( k = \frac{n-1}{2} \). This completes the proof.

To recap, as a function of \( k \), \( \binom{n}{k} \) keeps increasing strictly till \( k = \frac{n-1}{2} \). After that, the value of \( \binom{n}{k} \) remains unchanged as we move from \( k = \frac{n-1}{2} \) to \( k = \frac{n+1}{2} \), and so these are the largest binomial coefficients. Finally, as \( k \) goes beyond \( \frac{n+1}{2} \), \( \binom{n}{k} \) strictly decreases.

The even \( n \) case is virtually identical except for the fact that there is only one largest binomial coefficient, i.e. \( \binom{n}{\frac{n}{2}} \) (that is there is only one “maxima”). Since the statement and proof of the version of Theorem 1 for even \( n \) is virtually identical to that of the odd \( n \) case, we will omit it here (although you should try writing down the statement on your own to be sure you understand it).

### 3 Symmetry of binomial coefficients

We already discussed that the fact that binomial coefficients are “symmetric” about the middle. One way of stating this mathematically is

\[
\binom{n}{k} = \binom{n}{n-k}.
\]

We can rewrite the above statement so that it more accurately reflects the symmetry about the middle:

**Theorem 2.** If \( n \) is an even positive integer, then for all integers \( 1 \leq k \leq \frac{n}{2} \),

\[
\text{Binom}_n \left( \frac{n}{2} - k \right) = \text{Binom}_n \left( \frac{n}{2} + k \right),
\]

that is

\[
\binom{n/2}{k} = \binom{n/2}{n/2+k}.
\]

When \( n \) is an odd positive integer, then for all integers \( 1 \leq k \leq \frac{n-1}{2} \),

\[
\text{Binom}_n \left( \frac{n-1}{2} - k \right) = \text{Binom}_n \left( \frac{n+1}{2} + k \right),
\]

that is

\[
\binom{n-1/2}{k} = \binom{n+1/2}{n/2+k}.
\]
Proof. For the even case, let $1 \leq k \leq \frac{n}{2}$, and let $k' = \frac{n}{2} - k$. Then
\[
\binom{n}{\frac{n}{2} - k} = \binom{n}{k'} = \binom{n}{n - k'} = \binom{n}{n - (\frac{n}{2} - k)} = \binom{n}{\frac{n}{2} + k},
\]
where the second equality follows from the fact we stated above (before the statement of theorem): $\binom{n}{k} = \binom{n}{n - k}$.

The proof for the odd case is similar. \qed

We will now state one final property of binomial coefficients: when $n$ is odd, the sum of the first half of binomial coefficients is equal to the sum of the second half of binomial coefficients.

**Theorem 3.** Let $n$ be an odd positive integer. Then
\[
\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k}.
\]

Proof. Using the fact that $\binom{n}{k} = \binom{n}{n-k}$, we can rewrite the LHS as
\[
\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{n-k}.
\]

Let $k' = n - k$. Then, in the above sum, as $k$ increases from 0 to $\frac{k-1}{2}$, $k'$ will decreases from $n$ to $\frac{n+1}{2}$. Thus,
\[
\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{\frac{n+1}{2} + 1} + \binom{n}{\frac{n+1}{2}} = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k}.
\]

This completes the proof. \qed

For even $n$, there are an odd number of binomial coefficients in total (since there are $n+1$ of them in total), and so the notion of “first half of binomial coefficients” is not well-defined (which half is $\binom{n}{\frac{n}{2}}$ included in: the first or the second?), and so we have to slightly tweak the statement. This will be explored in your problem set for this week.