INSTRUCTIONS:

1. You have to solve 6 problems in 24 hours. To get full points for a problem, you must give details for all the steps involved in your solution AND arrive at the correct answer. Giving partial details or arriving at the wrong answer may result in a partial score.

2. You may leave your answer in terms of factorials, binomial coefficients, and/or powers of numbers.

3. Make sure you either submit a separate image for every problem or submit one PDF for the whole exam on Gradescope. If doing the latter, ensure that you mark the regions in the PDFs corresponding to each problem.

4. If you are suspected of cheating, you will be asked to appear for an oral exam. If you fail the oral exam, you will reported to the authorities.

5. Make sure you write your name and NetID in the space provided above.
Problem 1. [Blocking views - revisited (30 pts)]

There are 10 students in a class: \( s_1, \ldots, s_{10} \), such that if \( 1 \leq i < j \leq 10 \) then the height of student \( s_i \) is strictly less than that of student \( s_j \). We are making the 10 students stand in a row, one behind the other. We say an arrangement is blocking for \( s_i \) if \( s_i \)'s view is blocked, i.e. \( s_i \)'s line of sight will be blocked by a taller person. For example, the arrangement \( s_9 s_{10} s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 \) is blocking for both \( s_7 \) and \( s_8 \) since their line of sight will be blocked by \( s_{10} \). However, this arrangement is not blocking for \( s_9 \) or \( s_{10} \). Find the number of arrangements that are simultaneously blocking for \( s_1, s_2, \) and \( s_3 \), i.e. the number of arrangements in which the line of sight of all three is blocked.

Solution: The objects we are dealing with are arrangements of the students. Let \( U \) be the set of all arrangements. Let \( A_1 \) be all arrangements that are blocking for \( s_1 \), \( A_2 \) be all arrangements that are blocking for \( s_2 \), and \( A_3 \) be all arrangements that are blocking for \( s_3 \). We want to find

\[
|A_1 \cap A_2 \cap A_3|.
\]

Using the difference rule and De Morgan’s law,

\[
|A_1 \cap A_2 \cap A_3| = |U| - |\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3|.
\]

Let’s compute \( |\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3| \). Note that \( \overline{A}_i \) (for \( i \in \{1, 2, 3\} \)) is the set of all arrangements that are non-blocking for students \( s_i \). We can use inclusion-exclusion to compute \( |\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3|:\)

\[
|\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3| = |\overline{A}_1| + |\overline{A}_2| + |\overline{A}_3| - |\overline{A}_1 \cap \overline{A}_2| - |\overline{A}_1 \cap \overline{A}_3| - |\overline{A}_2 \cap \overline{A}_3|.
\]

- \( |\overline{A}_1| \) is the number of arrangements that are non-blocking for \( s_1 \). The only way an arrangement can be non-blocking for \( s_1 \) is if \( s_1 \) stands at the front of the line, and so there are \( 9! \) such arrangements (just permute the remaining 9 students behind \( s_1 \)). \( |\overline{A}_1| = 9! \)

- \( |\overline{A}_2| \) is the number of arrangements that are non-blocking for \( s_2 \). The only way an arrangement can be non-blocking for \( s_2 \) is if \( s_2 \) either stands at the front of the line, or stands in the second position only having \( s_1 \) in front of them. For the former case, there are \( 9! \) arrangements (just permute everyone behind \( s_2 \) in an arbitrary manner), and for the latter case there are \( 8! \) arrangements (\( s_1 \) and \( s_2 \) are in positions 1 and 2, and the 8 people behind them can be arranged in \( 8! \) ways). So, \( |\overline{A}_2| = 8! + 9! \).

- \( |\overline{A}_3| \) is the number of arrangements that are non-blocking for \( s_3 \). The only way an arrangement can be non-blocking for \( s_3 \) is if either \( s_3 \) appears in position 1, or in position 2 (with either \( s_1 \) or \( s_2 \) in front of them), or in position 3 with \( s_1, s_2 \) in front of them. For the first case (\( s_3 \) in position 1), there are \( 9! \) arrangements. For the second case, when \( s_3 \) is in position 2, there are \( \binom{2}{1} \) ways of deciding which one of \( s_1 \) or \( s_2 \) will stand in front of \( s_3 \), and \( 8! \) ways of permuting the 8 people behind \( s_3 \), and so the total number of arrangements is \( 2 \cdot 8! \). For the third case, when \( s_3 \) is in position 3, there are \( 2! \) ways of arranging \( s_1 \) and \( s_2 \) in front of \( s_3 \), and \( 7! \) ways of permuting the 7 people behind \( s_3 \), and so the number of arrangements is \( 2! \cdot 7! \). This means that

\[
|\overline{A}_3| = 9! + 2 \cdot 8! + 2 \cdot 7!.
\]

- \( |\overline{A}_1 \cap \overline{A}_2| \) is the number of arrangements that are non-blocking for \( s_1 \) and \( s_2 \). The only such arrangements are those in which \( s_1 \) is in position 1 and \( s_2 \) in position 2, and so we can only arrange the 8 people standing behind \( s_2 \). Thus, \( |\overline{A}_1 \cap \overline{A}_2| = 8! \).
• $|\overline{A_1} \cap \overline{A_3}|$ is the number of arrangements that are non-blocking for $s_1$ and $s_3$. The only such arrangements are those in which $s_1$ is in position 1, and $s_3$ is either in position 2, or in position 3 (with only $s_1$ and $s_2$ in front of them). For the former type ($s_1$ in position 1, $s_3$ in position 2), there are only $8!$ possibilities, and for the latter ($s_1$ in position 1, and $s_3$ in position 3 with $s_2$ in between them), there are $7!$ options since we can only permute the 7 people behind $s_3$. Thus,

$$|\overline{A_1} \cap \overline{A_3}| = 8! + 7!.$$ 

• $|\overline{A_2} \cap \overline{A_3}|$ is the number of arrangements that are non-blocking for $s_2$ and $s_3$. This is only possible if either (i) $s_2$ is in position 1 and $s_3$ in position 3, or (ii) $s_2$ is in position 1 and $s_3$ in position 3, and $s_1$ is between them, or (iii) $s_2$ is in position 2, $s_3$ in position 3, and $s_1$ in position 1. For (i), there are $8!$ such arrangements. For (ii), there are $7!$ such arrangements (permute everyone behind $s_3$. For (iii), there are also $7!$ such arrangements, and so

$$|\overline{A_2} \cap \overline{A_3}| = 8! + 2 \cdot 7!.$$ 

• Finally $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ are all arrangements that are non-blocking for all three: $s_1, s_2, s_3$. The only arrangements that satisfy this are ones in which $s_1, s_2, s_3$ are in positions 1, 2, 3 respectively. Since the remaining 7 people can be arranged in any way, there are $7!$ such arrangements, and so

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 7!$$

Thus,

$$|\overline{A_1} \cup \overline{A_2} \cup \overline{A_3}| = 9! + 8! + 9! + 9! + 2 \cdot 8! + 2 \cdot 7! - 8! - 8! - 7! - 8! - 2 \cdot 7! + 7!.$$ 

$$= 3 \cdot 9!.$$

Also note that $|U| = 10!$, and so

$$|A_1 \cap A_2 \cap A_3| = |U| - |\overline{A_1} \cup \overline{A_2} \cup \overline{A_3}| = 10! - 3 \cdot 9!.$$
Problem 2. [Social distancing (30 pts)]
There are 20 students in a class. The classroom in which the class is held has 90 seats in a single row facing the blackboard and the podium. We want to seat the 20 students in this row of 90 seats so that they maintain proper social distancing: there must be at least 2 empty seats between any two students. For example, if we only had 3 students and a row of 10 seats, then

\[ +s_2 + +s_1 + + + +s_3 \]

is a valid way of seating. Here \( s_1, s_2, s_3 \) are the three students, and \(+\) denotes an empty seat.

Find the number of ways of seating the 20 students among the 90 seats so that social distancing is followed, i.e. there are at least 2 empty seats between any two students.

Solution: We can construct a seating plan as follows, in two steps:

1. Choose which 20 out of the 90 seats will be occupied so that there are at least 2 unoccupied seats between every pair of occupied seats.

Let \( x_1 \) be the number of unoccupied seats to the left of the first occupied seat, \( x_2 \) be the number of unoccupied seats between the first occupied seat and second occupied seat, and in general, for \( 3 \leq i \leq 20 \), let \( x_i \) denote the number of unoccupied seats between the \((i-1)\)-th and \( i \)-th occupied seats. Let \( x_{21} \) denote the number of unoccupied seats to the right of the last occupied seats. Clearly, the total number of unoccupied seats is \( 90 - 20 = 70 \), and so

\[ x_1 + \ldots + x_{21} = 70, \]

where \( x_1, x_{21} \geq 0 \) (since it’s possible that there are no unoccupied seats to the left of the first or right of the last occupied seats) and \( x_2, \ldots, x_{20} \geq 2 \) (since we want there to be at least 2 unoccupied seats between any two occupied seats). The number of integer solutions to the above equation under these constraints can be found using the stars and bars method. Here \( n = 21, k = 70, r_1 = r_{21} = 0, \) and for all \( 2 \leq i \leq 20, r_i = 2 \). Recall that the number of integer solutions is given by

\[
\binom{k - \sum_{i} r_i + n - 1}{n - 1} = \binom{70 - 38 + 21 - 1}{21 - 1} = \binom{52}{20}.
\]

Thus, the number of ways of choosing 20 seats to occupy from the 90 ones so that social distancing is maintained is \( \binom{52}{20} \).

2. After we have decided which seats will be occupied, we need to decide who sits where. Having decided which seats will be occupied, there are 20! ways of arrange the students between those 20 seats.

Using the generalized product rule, the number of ways of seating that satisfies social distancing is \( \binom{52}{20}20! \).
**Problem 3.** [Mathematical expressions (20 pts)]

Consider the mathematical expression

\[(\num_1 + \num_2) \times (\num_3 - \num_4).\]

We have been asked to replace \(\num_1, \num_2, \num_3\) and \(\num_4\) by numbers between 1 and 35, i.e. numbers from the set \(\{1, \ldots, 35\}\), so that after substituting these numbers in place of \(\num_1, \num_2, \num_3\) and \(\num_4\), the evaluation of the expression using PEDMAS leads to an **odd positive integer**.

Find the number of ways in which this can be done. For example, one way of doing this is to use \(\num_1 = 2, \num_2 = 3, \num_3 = 5,\) and \(\num_4 = 2\). If we substitute these values we will get the expression

\[(2 + 3) \times (5 - 2) = 5 \times 3 = 15.\]

On the other hand, \(\num_1 = 1, \num_2 = 3, \num_3 = 5,\) and \(\num_4 = 2\) leads to an even integer, and \(\num_1 = 2, \num_2 = 3, \num_3 = 2,\) and \(\num_4 = 5\) leads to a negative odd integer, both of which are not allowed.

**Solution:** In order to get an odd positive integer, both \((\num_1 + \num_2)\) and \((\num_3 - \num_4)\) must be positive and odd. Let \(E_1\) denote the set of all 2-tuples \((\num_1, \num_2)\), where \(1 \leq \num_1, \num_2 \leq 35\), such that \(\num_1 + \num_2\) is odd and positive. Similarly, let \(E_2\) denote all 2-tuples \((\num_3, \num_4)\) where \(1 \leq \num_3, \num_4 \leq 35\), such that \(\num_3 - \num_4\) is positive and odd. Then the answer to this problem, by the product rule, is given by \(|E_1||E_2|\).

Let us first find \(|E_1|\). First notice that if \(\num_1, \num_2 \in \{1, \ldots, 35\}\) then \(\num_1 + \num_2\) is always positive. We now need ensure it’s odd. \(\num_1 + \num_2\) is odd if and only if either \(\num_1\) is odd and \(\num_2\) is even, or \(\num_1\) is even and \(\num_2\) is odd. In the case of the former, there are 18 possible options of odd numbers for \(\num_1\), and 17 options of even numbers for \(\num_2\), and so the total is \(18 \times 17\) in all. Similar, in the case of the latter (i.e., when \(\num_1\) is even and \(\num_2\) is odd), a similar argument shows that there are \(17 \times 18\) possibilities in all, and so

\[|E_1| = 2 \times 17 \times 18.\]

Let us now compute \(|E_2|\). For that define the set \(E\) as the set

\[E = \{(\num_3, \num_4) | 1 \leq \num_3, \num_4 \leq 35, \num_3 - \num_4\ is\ odd\}.\]

Clearly, \(E_2 \subset E\). Let us first find \(|E|\). This can be found in the same way as above: for every \((\num_3, \num_4) \in E\), either \(\num_3\) is odd and \(\num_4\) is even, or \(\num_3\) is even and \(\num_4\) is odd, and so \(|E| = 2 \times 17 \times 18\).

Let \(\bar{E}_2\) be the complement of \(E_2\) inside \(E\). Thus, since

\[E_2 = \{(\num_3, \num_4) \in E | \num_3 - \num_4\ is\ positive\},\]

it follows that

\[\bar{E}_2 = \{(\num_3, \num_4) \in E | \num_3 - \num_4\ is\ negative\}.\]

Also note that \(E_2 \cup \bar{E}_2 = E\), and that \(E_2\) and \(\bar{E}_2\) are disjoint, i.e., \(E_2, \bar{E}_2\) form a partition of \(E\). We will now show that there is a bijection between \(E_2\) and \(\bar{E}_2\). Consider the function defined as \(f((a, b)) = (b, a)\). Then given an \((a, b) \in E_2\), i.e., \(a - b\) is odd and positive, it is clear that \(f((a, b)) = (b, a)\) is odd and negative. Thus, \(f\) can be thought of as a function with domain \(E_2\) and
codomain $\bar{E}_2$. Furthermore, it’s not hard to observe that this function is injective and surjective since it simply swaps the two coordinates. Thus, $f$ is a bijection, and so

$$|\bar{E}_2| = |E_2|,$$

and recalling the fact that $E_2, \bar{E}_2$ form a partition of $E$, it follows that

$$|E_2| = |\bar{E}_2| = \frac{|E|}{2} = 17 \times 18.$$

The total number of solutions to the original expression is

$$|E_1||E_2| = 2 \times (17)^2 \times (18)^2.$$
Problem 4. [Palindromes (20 pts)]
Find the number of palindromes of length 72 consisting of 30 ‘a’s, 26 ‘b’s, and 16 ‘c’s that begin and end with an ‘a’.

Solution: By definition, the second half (i.e., the last 36 characters) of the palindrome will be essentially be the reversed version/mirror image of the first half (i.e., the first 36 characters), and so all we need to do is figure out how many possibilities are there for the first half. Also note that the number of ‘a’s in the first half is equal to the number of ‘a’s in the second half, and the same holds for the ‘b’s and ‘c’s. Furthermore, if the string starts with an ‘a’, then it will also end with an ‘a’ because of the mirror image property.

The above discussion suggests that the string formed by the first 36 letters will contain 15 ‘a’s, 13 ‘b’s, and 8 ‘c’s, and will begin with an ‘a’. If we can find the number of such strings then that will also be the number of palindromes of the desired form.

The letter has to be an ‘a’, so we are left with 14 ‘a’s, 13 ‘b’s, and 8 ‘c’s. To find the number of arrangements of these remaining 35 letters, we can use the BOOKKEEPER’s rule: the number of such arrangements is

$$\frac{35!}{14!13!8!}$$

Thus, this is also the number of possible palindromes of the desired form.
Problem 5. [Game of cards (10 pts)]
In how many ways can you choose 6 cards from a deck of 52 cards so that at least half of the chosen cards are spades?

Solution: Let $E$ be all subsets of the 52 cards of size 6 with at least 3 spades. For $3 \leq i \leq 6$, let $E_i \subset E$ denote all the sets in $E$ in which there are exactly $i$ spade cards. To find $|E_i|$, note that there are $\binom{13}{i}$ ways of choosing $i$ spade cards, and $\binom{39}{6-i}$ ways of choosing $6 - i$ non-spade cards, and so $|E_i| = \binom{13}{i} \binom{39}{6-i}$.

Also, observe that

$$E = E_3 \cup E_4 \cup E_5 \cup E_6,$$

and that $E_3, \ldots, E_6$ are disjoint sets. Thus, they form a partition of $E$. Using the sum rule, we can conclude that

$$|E| = \sum_{i=3}^{6} |E_i| = \sum_{i=1}^{6} \binom{13}{i} \binom{39}{6-i}.$$
Problem 6. [Coefficient (10 pts)]
Find the coefficient of $x^{49}$ in

$$(-4x + (6 + x)^{12})^5.$$ 

Solution: Using the binomial theorem,

$$(-4x + (6 + x)^{12})^5 = \sum_{i=0}^{n} \binom{5}{i} (-4x)^i (6 + x)^{12-i} = \sum_{i=0}^{n} \binom{5}{i} (-4)^i x^i (6 + x)^{60-12i}. $$

Note that if $i \geq 2$ then $x^i(6 + x)^{60-12i}$ cannot contain a term with $x^{49}$, since the maximum possible degree of $x$ we can get is $60 - 11i \leq 60 - 22 = 38$. Thus, the only terms in the above expansion that can contain $x^{49}$ are $i = 0$ and $i = 1$.

If $i = 0$, then the term corresponding to it from the above expression is $\binom{5}{0} (-4)^0 x^0 (6 + x)^{60} = \binom{5}{0} (x + 6)^{60}$, and the binomial theorem tells us that $(6 + x)^{60}$ will contain exactly one term corresponding to $x^{49}$: $\binom{60}{49} x^{49} 6^{11}$. Thus, the coefficient of $x^{49}$ in $\binom{5}{0} (-4)^0 x^0 (6 + x)^{60}$ is $\binom{5}{0} \binom{60}{49} 6^{11}$.

If $i = 1$, then the term corresponding to it from the above expression is $\binom{5}{1} (-4)^1 x^1 (6 + x)^{48} = \binom{5}{1} (-4) x(x + 6)^{48}$. To get $x^{49}$ from $x(x + 6)^{48}$, we need to look at the term that contains $x^{48}$ in $(x + 6)^{48}$, and this we know is $\binom{48}{48} x^{48} 6^{0}$ using the binomial theorem. Thus, the coefficient of $x^{49}$ in $\binom{5}{1} (-4)^1 x(x + 6)^{48}$ is $\binom{5}{1} (-4) \binom{48}{48}$.

Thus, the overall coefficient of $x^{49}$ is

$$\binom{5}{0} \binom{60}{49} 6^{11} + \binom{5}{1} (-4) \binom{48}{48}.$$