Problem 1

(Based on Problem 15.29 from the book)

In this problem, we will use the multinomial theorem to prove Fermat’s little theorem which says that for every prime number $p$ and any positive integer $n > 0$, $(n^p - n)$ is always a multiple of $p$. Let $p > 0$ be a prime number.

1. Using the definition of prime numbers, argue informally that if there is an integer $\alpha$ that can be written as a fraction
   \[ \alpha = \frac{a_1 \times a_2 \times \ldots \times a_n}{b_1 \times b_2 \times \ldots \times b_m}, \]
   where $a_1, \ldots, a_n > 0$ are integers such that at least one of them is a multiple of $p$ and $b_1, \ldots, b_m > 0$ are integers such that none of them is a multiple of $p$, then $\alpha$ must be a multiple of $p$.

2. Use the statement from part 1 to prove that the multinomial coefficient
   \[ \binom{p}{k_1, k_2, \ldots, k_n} \]
   is a multiple of $p$ if $k_1, \ldots, k_n > 0$ are all integers less than $p$.

3. Prove that the multinomial coefficient
   \[ \binom{p}{k_1, k_2, \ldots, k_n} \]
   is equal to 1 if exactly one of $k_1, k_2, \ldots, k_n$ is equal to $p$ (while the others are all equal to 0).

4. Use the results from part 1 and 2 along with the multinomial theorem to show that for all positive integers $x_1, \ldots, x_n > 0$,
   \[ (x_1 + x_2 + \ldots + x_n)^p - (x_1^p + x_2^p + \ldots + x_n^p) \]
   is a multiple of $p$.

5. Use the previous part to conclude that Fermat’s little theorem is true.

Problem 2

Find the coefficient of $x^{12}z^{18}$ in
\[ (-2 + (7 + x)^2 - (z - 6)^3)^{13}. \]

**Hint:** You will have to apply both the binomial and multinomial theorem for this problem. First let $X = (7 + x)^2$, $Z = -(z - 6)^3$, and $Y = -2$. Then the given polynomial can be written as $(Y + X + Z)^{13}$. Expand this using the multinomial theorem and then try to analyze which terms in this expansion can potentially contribute to the coefficient of $x^{12}z^{18}$. You might have to use the binomial theorem to figure that out.