Problem 1

Suppose there are 100 distinct objects which are to be distributed among 97 students. It is possible that some objects are not given to any of the students and are just discarded. In how many ways can you distribute the objects?

Solution: For each object, we have 98 choices: 97 choices for giving it to one of the students, and one choice for discarding it. The choices for the objects are independent, and so, using the product rule, the total number of ways of distributing is $98^{100}$.

Problem 2

Suppose 100 identical dice are rolled together. How many possible outcomes are there?

Hint: Think of the best way of “modeling” an outcome/communicating an outcome to a friend — what essential information do you need to convey to them which captures everything about the outcome?

Solution: We can “model” an outcome as follows: for each $i \in \{1, 2, \ldots, 6\}$, let $x_i$ be the number of dice (out of the 100 dice) that end up with the number $i$ on them. Then clearly, $x_1 + \ldots + x_6 = 100$. Thus, the number of outcomes is basically the number of non-negative integer solutions to $x_1 + \ldots + x_6 = 100$, which from part 3 is equal to $\binom{100+5}{5} = \binom{105}{5}$.

Problem 3

(Based on Problem 15.14(b) from the textbook)

Find the number of non-negative integer solutions to $x_1 + x_2 + \ldots + x_n = k$ where $n, k > 0$. By non-negative integer solutions, we mean values $x_1, \ldots, x_n \geq 0$ that satisfy the given equation.

Hint: Think of each of the $n$ variables as $n$ distinct persons and note that the problem is basically asking you to find the number of ways of distributing $k$ identical objects among $n$ distinct people.

Solution: There is a bijection between the set of non-negative integers solutions to $x_1 + \ldots + x_n = k$ and ways of distributing $k$ identical objects among $n$ distinct people. If we number the $n$ distinct people using numbers $1$ to $n$, then given a way of distributing $k$ objects among them, we can obtain a solution to the given equation by setting $x_i$ to be equal to the number of objects received by person $i$. Conversely, given a solution to the equation, we can come up with a way of distributing the $k$ objects among the $n$ people: we give person $i$ $x_i$ many objects. It is easy to verify that this is indeed a bijection between the two sets and so the numbers of solutions to the equation is equal to the number of ways of distributing $k$ identical objects among $n$ people, and we know the latter is equal to $\binom{k+n-1}{n-1}$. Thus, the number of non-negative integer solutions to $x_1 + \ldots + x_n = k$ is also the $\binom{k+n-1}{n-1}$.

Problem 4

100 identical looking balls are tossed arbitrarily into 20 distinct bins that are kept on the floor.

1. If the balls are tossed in a way that every ball falls into some bin, how many possible outcomes are there?

Solution: We can think of the 20 distinct bins as 20 different people, and the act of tossing the balls into the bins as distributing the 100 identical objects among the 20 people. Since every ball goes into some bin, this becomes
equivalent to the problem of distributing 100 identical objects among 20 distinct people, and the number of ways of doing the latter is \( \binom{119}{19} \).

2. If the balls are tossed in a way that some of the balls fall into a bin while some just land on the floor, how many possible outcomes are there?

   **Hint:** Think of the floor as a bin too; now there are 21 bins — does that make sense?

   **Solution:** Building on the hint, the problem is equivalent to the problem of distributing 100 identical objects among 20 distinct people, and so the number of possible outcomes is \( \binom{120}{20} \).