Problem 1

Three fair dice colored red, blue and green are rolled.

1. What will you choose the outcomes and sample space to be in this case?
   
   **Solution:** We can model an outcome using a 3-tuple \((r, b, g)\) whose first entry is the number that turns up on the red dice, the second is the number that turns up on the blue dice, and the third is the number that turns up on the green dice. All the entries of the 3-tuple are numbers between 1 and 6. The sample space is the 3-fold cartesian product of the set \(\{1, \ldots, 6\}\) with itself, i.e., \(\{1, \ldots, 6\}^3\). In other words, the set of all three tuples whose entries are numbers between 1 and 6.

2. What is the total number of events in the sample space? How many atomic events are there in the sample space?
   
   **Solution** Since \(\Omega = \{1, \ldots, 6\}^3\), it follows that \(|\Omega| = 6^3 = 216\). The number of events is \(|\text{pow}(\Omega)| = 2^{|\Omega|} = 2^{216}\).

3. What probability measure will you use to model the process of rolling the dice?
   
   **Solution:** Since all the dice are fair, we expect all outcomes to be equally likely, and so we use the uniform probability measure on \(\Omega\), i.e. for every outcome \(s \in \Omega\),
   
   \[ P(\{s\}) = \frac{1}{|\Omega|} = \frac{1}{216}. \]

4. What is the probability that exactly two of the dice roll the same number?
   
   **Solution:** Let \(E\) be the event that exactly two dice roll the same number. This event contains all 3-tuples in which exactly 2 of the entries are equal. We can construct such 3-tuples in the following way: we first decide which 2 out of the 3 entries have the same number — there are \(\binom{3}{2}\) ways of doing that. Then, we decide what number do these 2 entries have — there are 6 ways of deciding that. Finally, we have to decide what number does the remaining 1 entry have. This number has to be different from the number that the other two entries have, and so there are 5 choices for this step. Thus, we can see that \(|E| = \binom{3}{2} \times 6 \times 5 = 90\) using the generalized product rule. This means that
   
   \[ P(E) = \frac{|E|}{|\Omega|} = \frac{90}{216}. \]

5. What is the probability that all three dice roll distinct numbers?
   
   **Solution:** Let \(F\) be the event that all three dice roll distinct numbers. Then \(F\) is basically the set of all 3-tuples containing 3 distinct numbers. Using the generalized product rule, \(|F| = 6 \times 5 \times 4 = 120\). This means that
   
   \[ P(F) = \frac{|F|}{|\Omega|} = \frac{120}{216}. \]

6. What is the probability that at least two of the dice roll the same number?
   
   **Solution:** Let \(G\) denote the event that at least two of the dice have the same number on them. Then note that \(G\), i.e. the complement of \(G\), is the event that all three dice have distinct numbers, i.e.
   
   \[ G = F. \]
Using the difference method for sets,

\[ |G| = |\Omega| - |\overline{G}| = |\Omega| - |F| = 216 - 120 = 96, \]

and so

\[ P(G) = \frac{96}{216}. \]

Problem 2

You are at a casino table and are about to play a game. The rule of the game is as follows: at each step a fair coin is tossed and if it comes up heads you win $1, and if it comes up tails you lose $1. The game consists of 50 such steps.

1. What will you choose the outcomes and sample space to be in this case? What is the size of the sample space?

Solution: Outcomes can be modelled using binary strings of length 50, where 1 in position \( i \) indicates that step \( i \) was a heads (victory), and 0 indicates that step \( i \) was a tails (loss). \( \Omega \) is the set of all binary strings of length 50, and so \( |\Omega| = 2^{50} \).

2. What probability measure on the sample space will you use to model this random process?

Solution: Since we are using a fair coin, all outcomes should equally likely, and so we use the uniform measure, i.e. for every outcome \( s \in \Omega \),

\[ P(\{s\}) = \frac{1}{|\Omega|} = \frac{1}{2^{50}}. \]

Define your net gain to be the total amount of money won by you during the game minus the total amount of money lost by you during the game.

1. Consider the event

\( E_1 = \) Your net gain is zero at the end of the game.

How will you write \( E_1 \) as a subset of \( \Omega \)?

Solution: To have zero net gain, we should win and lose the same number of rounds, and so \( E_1 \) is the set containing all binary strings that have exactly 25 1s and 25 0s.

2. What is the probability that you end up with zero net gain?

Solution: This is just

\[ P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{\binom{50}{25}}{2^{50}}. \]

3. Consider the event

\( E_2 = \) Your net gain is positive at the end of the game.

How will you write \( E_2 \) as a subset of \( \Omega \)?

Solution: We would need strictly more wins than losses in order to have a positive net gain, and so \( E_2 \) is the set of all binary strings of length 50 that contain at least 26 1s.

4. What is the probability that you end with a positive net gain?

Solution: This is just

\[ P(E_2) = \frac{|E_2|}{|\Omega|} = \frac{\sum_{i=26}^{50} \binom{50}{i}}{2^{50}}. \]