Problem 1

Suppose a black dice and a white dice are rolled together. Use the partition method to find the number of possible outcomes where the number that turns up on the black dice is strictly larger than the number that turns up on the white dice.

Solution: Let $A$ be the set of all outcomes where the black dice turns up a greater number than the white dice. Define $A_i \subseteq A$, for $1 \leq i \leq 6$, as follows:

$$A_i = \{(b, w) | (b > w) \land (b = i)\}.$$  

You can verify that $A = A_1 \cup A_2 \cup \ldots A_6$ and that $A_1, \ldots, A_6$ are mutually disjoint sets. Thus, $A_1, \ldots, A_6$ form a partition of $A$ and using the partition rule we have that

$$|A| = \sum_{i=1}^{6} |A_i|.$$

Now note that

$$A_1 = \emptyset$$
$$A_2 = \{(2, 1)\}$$
$$A_3 = \{(3, 1), (3, 2)\}$$
$$\vdots$$
$$A_6 = \{(6, 1), (6, 2), \ldots, (6, 5)\},$$

and so $|A_1| = 0, |A_2| = 1, \ldots, |A_6| = 5$, and thus

$$|A| = \sum_{i=1}^{6} |A_i| = 0 + 1 + 2 + \ldots + 5 = 15$$

Problem 2

Suppose a black dice and a white dice are rolled together. Find the number of possible outcomes where the number that turns up on the black dice is at most the number that turns up on the white dice.

Solution: Let $A_{b \leq w}$ be the set $\{(b, w) | b \leq w, \ 1 \leq b, w \leq 6\}$. Let $U$ be the set of all possible outcomes, $A_{b = w}$ be the set of outcomes where the black and white dice have the same number, $A_{b > w}$ be the set of outcomes where the black dice has a greater number than the white dice and $A_{b < w}$ be the set of outcomes where the black dice has a smaller number than the white dice.

First observe that $A_{b < w}, A_{b > w}$ and $A_{b = w}$ are all mutually disjoint. Also,

$$U = A_{b < w} \cup A_{b = w} \cup A_{b > w}.$$  

We also know from the class that there is a bijection between $A_{b < w}$ and $A_{b > w}$, and so $|A_{b < w}| = |A_{b > w}|$. Applying the partition method on $U$ we get

$$|U| = 2 \cdot |A_{b < w}| + |A_{b = w}|.$$
Recall that $|A_{b=w}| = 6$ and $|U| = 36$ and so

$$|A_{b<w}| = \frac{|U| - |A_{b=w}|}{2} = \frac{36 - 6}{2} = 15.$$  

Now observe that $A_{b\leq w} = A_{b<w} \cup A_{b=w}$ and so using sum rule/partition method, we have that

$$|A_{b\leq w}| = |A_{b<w}| + |A_{b=w}| = 15 + 6 = 21.$$  

**Problem 3**

How many numbers are there between 1 and 100 that are either prime or multiples of 6?

**Solution:** Let $A$ be the set of multiples of 6 between 1 and 100 and $B$ the set of prime numbers between 1 and 100. Observe that $A \cap B = \emptyset$ and we are interested in finding $|A \cup B|$. Using sum rule, it follows that

$$|A| = \text{floor} \left( \frac{100}{6} \right) = 16.$$  

There is really no pattern to prime numbers so we can try and enumerate the ones between 1 and 100:

$$B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, \ldots, 97 \}.$$  

You can verify that $|B| = 25$ and so

$$|A \cup B| = 25 + 16 = 41.$$  

**Problem 4**

Suppose a black dice and a white dice and rolled together. Find the number of possible outcomes where the product of the numbers on the two dice is odd.

**Hint:** Let $A$ be the set of all outcomes where the product of the two numbers is odd. Try to partition $A$ into parts based on the number on the black dice.

**Solution:** Let $A$ be the set of all outcomes where the product is odd. Define $A_i \subseteq A$, for $1 \leq i \leq 6$, to be the set defined as $A_i = \{(b, w) | b = i, b \cdot w \text{ is odd}\}$. Clearly, $A_1, \ldots, A_6$ form a partition of $A$ (i.e., their union is equal to $A$ and they are mutually disjoint), and so

$$|A| = \sum_{i=1}^{6} |A_i|.$$  

Let us see what these sets really are. First note that $A_2 = A_4 = A_6 = \emptyset$ because the outcomes in these sets always lead to the product being even (why?). Similarly, $|A_1| = |A_3| = |A_5| = 3$ because in each case in order to make sure the product is odd the white dice cannot equal to 2, 4 or 6, leaving only three possibilities: 1, 3, 5. Thus,

$$|A| = 3 + 0 + 3 + 0 + 3 + 0 = 9.$$  

**Problem 5**

Suppose a black dice and a white dice and rolled together. Find the number of possible outcomes where the sum of the numbers on the two dice is not divisible by 7.

**Hint:** Difference method

**Solution:** Let $A$ be the set of outcomes where the sum is not divisible by 7 and let $U$ be the set of all possible outcomes. Then $A = U - A$ is the set all outcomes where the sum is divisible by 7. Also, it is easy to list out the elements of $U - A$:

$$U - A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$  

and so $|U - A| = 6$. We know that $|U| = 36$, and so

$$|A| = |U| - |U - A| = 36 - 6 = 30.$$  

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Problem 6

Use the partition method to show that the total number of possible outcomes when three dice are rolled (a white, a black, and a red dice) is 216.

**Hint:** Use a 3-tuple \((b, r, w)\) to model an outcome. The set of all outcomes, then, is \(A \times A \times A\), where \(A = \{1, \ldots, 6\}\). Now partition \(A \times A \times A\) into 6 parts: \(A_1, \ldots, A_6\) such that \(A_i\) is the set of all outcomes where the black dice turns up the number \(i\). What is \(|A_i|\)? Use that along with the sum rule to get the answer.

**Solution:** Let us continue the strategy mentioned in the hint. Note that \(A_i\) is the set of all outcomes where the black dice turns up the number \(i\). How many such outcomes are there? The red and white dice can be any value between 1 and 6, and in fact it is easy to see that there is a bijection between \(A_i\) and the set of all possible outcomes if we were to only roll the white and red dice (do you see why?). Since we know that the number of outcomes for the latter is 36, it follows that \(|A_i| = 36\). Since \(A_1, \ldots, A_6\) partition \(A \times A \times A\), we get that

\[
|A \times A \times A| = \sum_{i=1}^{6} |A_i| = 6 \cdot 36 = 216.
\]

Problem 7

A binary string is called balanced if it has equal number of ones and zeros. Let \(S\) be the set of balanced binary strings of length 14, and let \(P\) be the set of paths from \(A\) to \(B\) in the grid below that use only downward and rightward steps (i.e., a path can never go up, left, or diagonal; only down or right). Show that \(|P| = |S|\).

**Solution:** We can setup a bijection between \(P\) and \(S\) as follows: any path \(p\) from \(A\) to \(B\) that only involves going down or right (and never up or left) can be encoded as a binary string, i.e. a sequence of 1s and 0s; whenever the path has a down move we put in a 0, and whenever it has a right move we put in a 1. Observe that any path from \(A\) to \(B\) must involve exactly 7 right moves and 7 down moves, and thus the above encoding procedure converts each path into a binary string of length 14 containing equal number of 0s and 1s, i.e. a balanced binary string.

Conversely, any balanced binary string can be “decoded” to get back a path from \(A\) to \(B\) that involves only down and right moves by simply reversing the above procedure. This shows that the encoding/decoding procedure is a bijection between \(P\) and \(S\) and so \(|P| = |S|\).