Problem 1

A valid password must satisfy the following conditions:

1. It only contains lowercase characters $a$ to $z$ and digits 0 to 9.
2. It contains exactly 6 characters.
3. It contains at least one digit.

Find the number of valid passwords.

**Hint:** Difference method.

**Solution:** Let $U$ be the set of all 6 letter strings you can form using letters $a$ to $z$ and digits 0 to 9. Using the product rule, $|U| = 36^6$.

Let $A$ be the set of valid passwords. Then $U - A$ is the set of all strings of length 6 consisting only of letter $a$ to $z$ (i.e., no digits!). Again, using product rule, $|U - A| = 26^6$. It follows that

$$|A| = |U| - |U - A| = 36^6 - 26^6.$$

Problem 2

Do problem 15.5 in the textbook.

**Solution:** Part (a) The standard plate numbers can be represented as $A^3 \times D^3$, the vanity plates can be represented as $A^5$, and the big shot plates can be represented as $(A \cup D)^2$. Thus,

$$L = (A^3 \times D^3) \cup A^5 \cup (A \cup D)^2$$

Part (b) We can apply the partition method to conclude that

$$|L| = |(A^3 \times D^3)| + |A^5| + |(A \cup D)^2|.$$ 

Using the product rule, $|A^5| = |A|^5 = 26^5$. Also, using the sum rule, $|(A \cup D)| = |A| + |D| = 36$, and so $|(A \cup D)^2| = 36^2$ using the product rule.

Finally, note that the product rule gives us $|A^3| = 26^3$ and $|D^3| = 10^3$, and so $|(A^3 \times D^3)| = (26^3)(10^3)$. Thus,

$$|L| = (26^3)(10^3) + 26^5 + 36^2.$$ 

Problem 3

Do problem 15.6(a) in the textbook.

**Hint:** Forget about the last number in the range, i.e. 10^9, for the time being. So we are only looking at numbers from 1 to 999999999. We can think of all these numbers as strings of length 9 made from digits 0 to 9: 1 can be thought of as 000000001, 10 as 000000010, and so on. We are interested in finding all such strings of length 9 that contain the digit 1. Try to instead find the strings that don’t contain the digit 1. Also, watch out for 000000000 in your counting
and include $10^9 = 1000000000$ at the end of your calculations.

**Solutions:** As mentioned in the hint all the number from 1 to 999999999 can be thought of as 9-digit sequences made using digits 0 to 9 (excluding the 000000000 sequence). Let $S$ be the set of all such sequences. Using the product rule, we know that the number of sequences that can be made using digits 0 to 9 are $10^9$. This means that $|S| = 10^9 - 1$ since we subtract one to exclude the all zeros sequence.

Let $A \subset S$ be those sequences in $S$ that contain the digit 1. Then $S - A$ contains those sequences from $S$ that do not contain the digit 1, i.e. they are made only using the digits 0, 2, 3, . . . , 9. The number of sequences of length 9 that can be made using digits 0, 2, . . . , 9 are $9^9$ which means that $|S - A| = 9^9 - 1$ since we must exclude the all zeros sequence. Using the difference method,

$$|A| = |S| - |S - A|,$$

and so

$$|A| = 10^9 - 1 - (9^9 - 1) = 10^9 - 9^9.$$

Now so far we’ve excluded the number 1000000000 from our conversation and clearly it contains the digit 1, so we include it in our count and the final answer is $10^9 - 9^9 + 1$.

**Problem 4**

Do problem 15.4 in the textbook using the bijection between subsets and binary strings we discussed in class.

**Solution: Part (a)** Recall the bijection between subsets of $\{x_1, \ldots, x_6\}$ and binary strings of length 6. Let $S_1$ be the set of all subsets that contain $x_1$ and let $T_1$ be the set of all binary strings of length 6 that begin with a 1. It is not hard to see that there is a bijection between $T_1$ and $S_1$ (this is the same bijection that’s there between all subsets and all binary strings) and so $|S_1| = |T_1|$.

Noting that $|T_1| = 2^6$ using the product rule (Convince yourself!) we can conclude that $|S_1| = 2^6$.

**Part (b):** Let $S_2$ be the set of all subsets that contain $x_2$ and $x_3$ but not $x_6$. Let $T_2$ be the set of all binary strings that have a 1 in position 2 and 3 (from the left) and a 0 in position 6 (again, from the left). Again, using the same idea as in part (a), we can conclude that there is a bijection between $S_2$ and $T_2$ and so $|S_2| = |T_2|$.

To compute $|T_2|$, note that we are counting the number of 6-bit strings three of whose bits have already been fixed (namely the bits in position 1, 2 and 3). Thus, we are only allowed to choose bits (either 0 or 1) for the remaining 3 positions and we can do that independently across the positions. Thus, using the product rule $|T_2| = 2^3$, and so $|S_2| = 2^3$.

**Problem 5**

A park at Disney World has 30 different attractions. You are planning to visit each attraction exactly once but are unsure about the order in which you want to visit them. How many possible ways are there to visit each of the 30 attractions exactly once? (Basically, what is the number of different possible “orders”)?

**Solution:** Let assign a unique number between 1 and 30 to each of the 30 attractions. Each “order” can be modelled as a sequence of length 30 in which each number between 1 and 30 appears exactly once. This is basically just a permutation/arrangement of the 30 numbers, and so the number of permutations/arrangements/orders is

$$30! = 30 \times 29 \times \ldots \times 2 \times 1$$