Problem 1

Recall that during lecture we proved that
\[ \sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} = \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k} = 2^{n-1}, \]
which can be interpreted to mean that the number of odd size subsets of \{1, \ldots, n\} is the same as the number of even size subsets of \{1, \ldots, n\}, and both are equal to 2^{n-1}. In this problem, we will give a combinatorial proof of this statement. By this we mean that we will use the bijection method to show that the number of even size subsets of \{1, \ldots, n\} is the same as the number of odd size subsets of \{1, \ldots, n\}, and then use that to conclude the above equation. We will then use the conclusion to say something about binary strings with an odd/even number of ones in them.

1. Let \( S_{\text{odd}} \) and \( S_{\text{even}} \) denote the set of odd and the set of even size subsets of \{1, \ldots, n\}, respectively. Let us define the following transformation that takes as input a subset \( S \) of \{1, \ldots, n\} and maps/transforms it into the set \( S \oplus \{1\} \). Argue that this transformation can be used to define a function \( f : S_{\text{odd}} \rightarrow S_{\text{even}} \) that, given an odd size subset from the domain as input, maps it to an even size subset in the codomain.

2. Argue that \( f \) as defined in the previous part is a bijection.

3. Use the previous part to conclude that
\[ \sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} = \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k}. \]

**Hint:** Bijection method.

4. Finally, use the previous part, along with Theorem 3 from the notes for lecture 8 to conclude that
\[ \sum_{k \leq n \text{ and } k \text{ is even}} \binom{n}{k} = \sum_{k \leq n \text{ and } k \text{ is odd}} \binom{n}{k} = 2^{n-1}. \]

**Hint:** If \( x + y = A \) and \( x = y \) then \( x = y = \frac{A}{2} \).

5. Argue that the number of binary strings of length \( n \) with an odd number of ones in them is \( 2^{n-1} \). Similarly, argue that the number of binary strings of length \( n \) with an even number of ones in them is also \( 2^{n-1} \).

Problem 2

Do Problem 15.28 from the textbook.

**Hint:** Use the binomial theorem.

For part (b), let \( X = 3x \) and \( Y = 2y \). Then \( (3x + 2y)^n = (X + Y)^n \). Use the binomial theorem to “expand” \( (X + Y)^n \). Put back \( X = 3x \) and \( Y = 2y \) in the expansion, simplify, and then infer the coefficient of \( x^8y^9 \).

For part (c), use a similar strategy as part (b). Then it is not hard to see that the only monomials that appear are of the form \( (a^2)^k(b^3)^{5-k} \) where \( 0 \leq k \leq 5 \). Use this information to find the coefficient of \( a^6b^6 \).