Problem 1

Recall that during lecture we proved that for odd values of \( n \)
\[
\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}.
\]
which can be interpreted to mean that the number of subsets of \( \{1, \ldots, n\} \) of size (strictly) less than \( \frac{n}{2} \) is the same as the number of subsets of \( \{1, \ldots, n\} \) of size (strictly) greater than \( \frac{n}{2} \). In this problem, we will give a combinatorial proof of this statement. By this we mean that we will use the bijection method to prove the statement. We will then use the conclusion to say something about binary strings of length \( n \) with less than \( \frac{n}{2} \) ones.

1. Let \( S_{< \frac{n}{2}} \) denote the set of subsets of \( \{1, \ldots, n\} \) of size (strictly) less than \( \frac{n}{2} \) and \( S_{> \frac{n}{2}} \) denote the set of subsets of \( \{1, \ldots, n\} \) of size (strictly) greater than \( \frac{n}{2} \). Let us define the following transformation that takes as input a subset \( S \) of \( \{1, \ldots, n\} \) and maps/transforms it into the set \( \bar{S} \), i.e. the complement of \( S \) inside \( \{1, \ldots, n\} \). Argue that this transformation can be used to define a function \( f: S_{< \frac{n}{2}} \rightarrow S_{> \frac{n}{2}} \) that, given a subset of size \( < \frac{n}{2} \) from the domain as input, maps it to a subset of size \( > \frac{n}{2} \) in the codomain.

**Solution:** Let \( f(X) = \bar{X} \) where \( X \) is a subset of \( \{1, \ldots, n\} \). Clearly, if \( X \in S_{< \frac{n}{2}} \) then \( f(X) = \bar{X} \) will have \( n - |X| > \frac{n}{2} \) elements, and so \( f(X) \in S_{> \frac{n}{2}} \). Thus, \( f \) is well defined as a function with domain \( S_{< \frac{n}{2}} \) and codomain \( S_{> \frac{n}{2}} \).

2. Argue that \( f \) as defined in the previous part is a bijection.

**Solution:** It is easy to see that \( f \) is injective since if \( X, Y \in S_{< \frac{n}{2}} \) such that \( X \neq Y \) then it follows from properties of the complement operator that \( \bar{X} \neq \bar{Y} \) which then implies that \( f(X) \neq f(Y) \).

To show that \( f \) is surjective, let \( A \in S_{> \frac{n}{2}} \) be an arbitrary set in the codomain of \( f \). Let \( X = \bar{A} \). Then since \( |A| < \frac{n}{2} \), it must be the case that \( |X| = |\bar{A}| > \frac{n}{2} \) and so \( X \in S_{> \frac{n}{2}} \). Thus, \( X \) is in the domain. Furthermore,
\[
f(X) = \bar{X} = \bar{\bar{A}} = A,
\]
and so we have shown that there is an element \( X \) in the domain such that \( f(X) = A \). Since \( A \) was arbitrary, we can conclude that \( f \) is surjective, and hence bijective.

3. Use the previous part to conclude that
\[
\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}.
\]

**Hint:** Bijection method.

**Solution:** Observe that
\[
S_{< \frac{n}{2}} = \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}
\]
and
\[
S_{> \frac{n}{2}} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}.
\]
Since there is a bijection between $S_{< \frac{n}{2}}$ and $S_{> \frac{n}{2}}$, it follows that $|S_{< \frac{n}{2}}| = |S_{> \frac{n}{2}}|$ and so

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}.$$ 

4. Finally, use the previous part, along with Theorem 3 from the notes for lecture 8 to conclude that

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} = 2^{n-1}.$$ 

**Hint:** If $x + y = A$ and $x = y$ then $x = y = \frac{A}{2}$.

**Solution:** Theorem 3 from lecture 8 tells us that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$ 

Because $n$ is odd, we can rewrite this as

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} + \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} = 2^n.$$ 

Let $x = \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$ and $y = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}$. Then we know that $x = y$ and also $x + y = 2^n$ and so $x = y = 2^{n-1}$. Thus,

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} = 2^{n-1}.$$ 

5. Argue that the number of binary strings of length $n$ which have less than $\frac{n}{2}$ ones in them is also $2^{n-1}$. Similarly, argue that the number of binary strings of length $n$ with more than $\frac{n}{2}$ ones in them is also $2^{n-1}$.

**Solution:** Follows from the bijection between binary strings of length $n$ and set of subsets of $\{1, \ldots, n\}$.

6. Do the equations in parts 3 and 4 still hold if $n$ is even? Why or why not?

**Solution:** Let us assume now that $n$ is even. This means that there is exactly one middle binomial coefficient $\binom{n}{\frac{n}{2}}$. Let $S_{< \frac{n}{2}}$ and $S_{> \frac{n}{2}}$ be defined as before. Observe that $f$ defined as $f(X) = \bar{X}$ still remains a valid function from $S_{< \frac{n}{2}}$ to $S_{> \frac{n}{2}}$ since if $|X| < \frac{n}{2}$ then $|\bar{X}| = n - |X| > \frac{n}{2}$ regardless of whether $n$ is even or odd. Furthermore, $f$ defined as above is also a bijection between $S_{< \frac{n}{2}}$ and $S_{> \frac{n}{2}}$ — the same proof we used in part 3 works out. It also remains true that

$$S_{< \frac{n}{2}} = \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$$ 

and

$$S_{> \frac{n}{2}} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k},$$

and so the statement of part 3 is also true, i.e.

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k}.$$ 

However, the statement in part 4 is no longer true, i.e. summing the binomial coefficients from $k = 00$ to $k < \frac{n}{2}$ is no longer $2^{n-1}$ (likewise for the sum from $k > \frac{n}{2}$ to $k = n$). This is mainly because there is a middle binomial coefficient $\binom{n}{\frac{n}{2}}$ which is not participating in these sums.

Formally speaking, suppose for the sake of contradiction that this were true, i.e.

$$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} = \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} = 2^{n-1}.$$ 

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We know from Theorem 3 of lecture 8 that
\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n, \]
which can be further rewritten as
\[ \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} + \binom{n}{\frac{n}{2}} + \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} = 2^n. \]

Now, if the first and third term in the sum on the LHS were both equal to \( 2^{n-1} \) we would get
\[ 2^{n-1} + \binom{n}{\frac{n}{2}} + 2^{n-1} = 2^n \]
\[ \implies \binom{n}{\frac{n}{2}} + 2^n = 2^n \]
\[ \implies \binom{n}{\frac{n}{2}} = 0, \]
which is a contradiction since the \( \binom{n}{\frac{n}{2}} \) can never be less than 1 (even when \( n = 0! \)). Thus, our initial assumption must have been wrong, and so the statement of part 4 is not true when \( n \) is even.

**Trivia:** It turns out in fact that \( \binom{n}{\frac{n}{2}} \) is always approximately around \( \frac{2^n}{\sqrt{n}} \) which would then mean that both \( \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} \) and \( \sum_{\frac{n}{2} < k \leq n} \binom{n}{k} \) are approximately \( 2^{n-1} - \frac{2^{n-1}}{\sqrt{n}} \) which is less than \( 2^{n-1} \).

**Problem 2**

Let \( 1 \leq k < n \) be integers. Consider the equation
\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \]
This is called *Pascal’s identity.*

1. Use the formula for binomial coefficients along with algebra to show that Pascal’s identity is true.

**Solution:** We can use the formula for binomial coefficients and apply it to the RHS:
\[
\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} + \frac{(n-1)!}{k!(n-1-k)!} \\
= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left( \frac{1}{n-k} + \frac{1}{k} \right) \\
= \frac{(n-1)!}{(k-1)!(n-k-1)!} \left( \frac{n}{k(n-k)} \right) \\
= \frac{n(n-1)!}{k(k-1)! \times (n-k)(n-k-1)!} \\
= \frac{n!}{k!(n-k)!} = \binom{n}{k},
\]
which is the LHS, and so the identity is true.
2. Let $S$ be the set of all binary strings of length $n$ with exactly $k$ ones. Let $A \subseteq S$ be the subset of $S$ that consists of strings in $S$ whose first bit (from the left) is set to 1, and let $B \subseteq S$ be the subset of $S$ that consists of strings in $S$ whose first bit is set to 0. Find the sizes of $S$, $A$, and $B$ in terms of binomial coefficients.

**Solution:** Clearly, $|S| = \binom{n}{k}$. To form a string in $B$, we must choose $k$ locations for the $k$ ones from the last $n-1$ positions (we are excluding the first position from the left, since it must be zero), and so $|B| = \binom{n-1}{k}$ since there are $\binom{n-1}{k}$ ways of doing this.

Similarly, to construct a string in $A$, we only need to pick $k-1$ positions for $k-1$ ones from the last $n-1$ positions (again, excluding the first position since it must be 1). We only need to place $k-1$ ones because a string in $A$ has $k$ ones in total, and the first bit must be 1 by definition of $A$, so we are left $k-1$ ones to place. The number of ways of doing this process is $\binom{n-1}{k-1}$, and so $|A| = \binom{n-1}{k-1}.$

3. Use the previous part to prove that Pascal’s identity is true.

**Hint:** Observe that $A$ and $B$ form a partition of $S$.

**Solution:** Every string in $S$ has either a 1 in its first position or a 0 in its first position, and so $A \cup B = S$, and also $A \cap B = \emptyset$ (why?). Thus, $A$ and $B$ form a partition of $S$, and so using the sum rule

$$|S| = |A| + |B|$$

$$\implies \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Problem 3**

Find the coefficients of $x^8$ in $x(1 − 3x)^5(7 + 5x)^6$.

**Hint:** Note that there are multiple ways of forming $x^8$ when multiplying the terms out, and all of those ways will “contribute” to the coefficient of $x^8$.

**Solution:** Finding the coefficient of $x^8$ in the given polynomial is the same as finding the coefficient of $x^7$ in the polynomial $(1 − 3x)^5(7 + 5x)^6$. We can use the binomial theorem to write

$$(1 − 3x)^5(7 + 5x)^6 = \left( \sum_{k=0}^{5} \binom{5}{k} 1^k (-3x)^{5-k} \right) \left( \sum_{j=0}^{6} \binom{6}{j} 7^j (5x)^{6-j} \right)$$

If we multiply out the two sums in the RHS, a generic term in the product will look like

$$\binom{5}{k} \binom{6}{j} (-3x)^{5-k} 7^j (5x)^{6-j},$$

which on rearranging terms looks like

$$\binom{5}{k} \binom{6}{j} (-3)^{5-k} 7^j 5^{6-j} x^{5-k+6-j}.$$ 

The terms that will give us $x^7$ are those terms where $5-k+6-j = 7$, or $k+j = 4$, where $0 \leq k \leq 5$ and $0 \leq j \leq 6$.

Here are the solutions to $k+j = 4$ under the constraints on $k, j$:

- $k = 0, j = 4$
- $k = 1, j = 3$
- $k = 2, j = 2$
- $k = 3, j = 1$
- $k = 4, j = 0$.

Thus, the coefficient of $x^7$ in the product (and hence $x^8$ in the original expression) will be the sum of the coefficients corresponding to each of the above four cases:

$$-3 \cdot 5^6 \binom{5}{1} \binom{6}{6} + 3^2 \cdot 7 \cdot 5^5 \binom{5}{2} \binom{6}{5} - 3^3 \cdot 7^2 \cdot 5^4 \binom{5}{3} \binom{6}{4} + 3^4 \cdot 7^3 \cdot 5^3 \binom{5}{4} \binom{6}{3} - 3^5 \cdot 7^4 \cdot 5^2 \binom{5}{5} \binom{6}{2}.$$