Quiz II (Solutions) (CS 206 - Spring 2020) [2 problems, 20 pts, 20 mins]

Name: 
NetID: 

This is an open notes quiz; you may use any printed or hand-written material. The use of any electronic devices during the quiz or other attempts at cheating will have dire consequences.

1. (10 pts) Let $S = \{1, 2, \ldots, 20\}$ and $T = \{1, 2, \ldots, 10\}$. Let $\mathcal{F}$ be the set of all functions $f : S \to T$ such that for all $1 \leq i \leq 10$,

$$f(i) \leq 5,$$

and for all $11 \leq i \leq 20$,

$$f(i) \geq 6.$$

Find $|\mathcal{F}|$. **Explain the main steps of your solution in a brief manner.**

**Solution:** A function $f : S \to T$ can be specified by choosing values for $f(1), f(2), \ldots, f(20)$. We are interested in only those functions in which $f(1), f(2), \ldots, f(10)$ are all between 1 and 5 (inclusive), and $f(11), f(12), f(13), \ldots, f(20)$ are all values between 6 and 10 (inclusive). This means there are 5 choices for $f(1)$ (either 1, 2, 3, 4, or 5), 5 choices for $f(2)$, and so on, up until $f(10)$ for which we have 5 choices. Similarly we have 5 choices for $f(11)$ (either 6, 7, 8, 9, or 10), 5 choices for $f(12)$, and so on up until $f(20)$. In short, we have 5 choices for each of $f(1), f(2), \ldots, f(20)$ and all these choices are independent, and so using the product rule the number of such functions is $5^{20}$. 
2. (10 pts) Suppose there are 10 girls and 10 boys in a group (You may assume no two people in this group are identical). We want to make all of them sit in a single row of seats. The left-most seat in the row is numbered 1, the second-from-the-left seat is numbered 2, and so on, with the rightmost seat being numbered 20. In how many ways can we seat the 10 girls and 10 boys so that every girl is seated in an even-numbered seat and every boy is seated in an odd numbered seat. **Explain the main steps of your solution in a brief manner.**

**Solution:** For the first seat, there are 10 choices (any of the 10 boys). For the second seat, there are 10 choices (any of the 10 girls). Having fixed the first two choices, there are 9 choices for the third seat (any of the 9 remaining boys), and similarly there are 9 choices for the fourth seat. Having fixed the first four choices, we have 8 choices for the fifth seat (any of the remaining boys; there are 8 of them) and 8 choices for the sixth seat (any of the remaining 8 girls). We can proceed similarly until we come to the last two seats (the rightmost two seats), where, having fixed all the previous choices, we will have 1 choice each for the two seats (numbered 19, 20). Thus, using the generalized product rule, the number of ways of seating that respects the constraints is

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10 \times 10 \times 9 \times 9 \times 8 \times 8 \times \ldots \times 2 \times 2 \times 1 \times 1 = 10! \times 10!.
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