Quiz III (Solutions) (CS 206 - Spring 2020)

[2 problems, 20 pts, 20 minutes]

Name:
NetID:

This is an open notes quiz; you may use any printed or hand-written material. The use of any electronic devices during the quiz or other attempts at cheating will have dire consequences.

1. (5 × 2 pts each = 10 pts) State which of the following are true/false. You don’t need to explain/prove your answers.

(a) \( \binom{100}{33} > \binom{100}{80} \)

True. \( \binom{100}{80} = \binom{100}{20} < \binom{100}{33} \) since the binomial coefficients are increasing with \( k \) up till \( k = \frac{n}{2} \) which is 50 in this case.

(b) \( \frac{\binom{21}{0} + \binom{21}{1} + \ldots + \binom{21}{10}}{2^{21}} > \frac{1}{2} \)

False. When \( n \) is odd, \( \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{\frac{n-1}{2}} = 2^{n-1} \), and so the numerator is \( 2^{21-1} = 2^{20} \) and so \( \frac{2^{20}}{2^{21}} = \frac{1}{2} \).

(c) \( \frac{\binom{18}{0} + \binom{18}{1} + \ldots + \binom{18}{9}}{2^{18}} = \frac{1}{2} \)

False. Assume for the sake of contradiction that the fraction is equal to \( \frac{1}{2} \). Then it must be the case that the numerator of the fraction is equal to \( 2^{17} \), i.e.

\[
\binom{18}{0} + \ldots + \binom{18}{8} + \binom{18}{9} = 2^{17}.
\]

But since the sum of all binomial coefficients, i.e, \( \binom{n}{0} + \ldots + \binom{n}{n} \), is equal to \( 2^n \) it follows that

\[
\binom{18}{10} + \ldots + \binom{18}{18} = 2^{18} - 2^{17} = 2^{17}.
\]

When \( n \) is even, \( \binom{n}{0} + \ldots + \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n}{2} + 1} + \ldots + \binom{n}{n} \) and so

\[
\binom{18}{0} + \binom{18}{1} + \ldots + \binom{18}{8} = \binom{18}{10} + \binom{18}{1} + \ldots + \binom{18}{18}.
\]

Combining all three equations, we get

\[
\binom{18}{0} + \ldots + \binom{18}{8} + \binom{18}{9} = 2^{17}
\]

\[
\implies \binom{18}{10} + \ldots + \binom{18}{18} + \binom{18}{9} = 2^{17}
\]

\[
\implies 2^{17} + \binom{18}{9} = 2^{17}
\]
\[ \binom{18}{9} = 0, \]

which is a contradiction to the fact that \( \binom{18}{9} = \frac{18!}{9!9!} > 0 \). Thus, the given fraction cannot be equal to \( \frac{1}{2} \).

(d) \( \binom{1001}{251} = \binom{999}{249} + \binom{999}{250} + \binom{1000}{251} \)

True. Use the formula \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \) twice. So \( \binom{999}{249} + \binom{999}{250} = \binom{1000}{250} \) which when added to \( \binom{1000}{251} \) gives the LHS.

(e) Let \( n \) be an odd positive integer, \( A \) be the set of all binary strings of length \( n \) with an odd number of ones in them, and \( B \) be the set of all binary strings of length \( n \) which contain strictly less than \( \frac{n}{2} \) ones in them. Then \( |A| = |B| \).

True. Recall that both \( |A| \) and \( |B| \) are equal to \( 2^n - 1 \) when \( n \) is odd.

2. (10 pts) Find the coefficient of \( x^7 \) in \( (1 - x)(1 + 3x)^7 + 2(1 + x^3)(7 - 3x^2)^4 \).

Solution: We can write

\[ (1 - x)(1 + 3x)^7 + 2(1 + x^3)(7 - 3x^2)^4 = (1 + 3x)^7 + (-x)(1 + 3x)^7 + 2(7 - 3x^2)^4 + 2x^3(7 - 3x^2)^4. \]

It suffices to find the the coefficients of \( x^7 \) in each of the four terms in the sum in the RHS and then adding them up. The coefficient of \( x^7 \) in \( (1 + 3x)^7 \) is \( \binom{7}{7}3^7 \). The coefficient of \( x^7 \) in \( (-x)(1 + 3x)^7 \) is the negative of the coefficient of \( x^6 \) in \( (1 + 3x)^7 \) which is \(-\binom{7}{6}3^6\).

As for \( 2(7 - 3x^2)^4 \), note that

\[ 2(7 - 3x^2)^4 = \sum_{k=0}^{4} \binom{4}{k}7^k(-3x^2)^{4-k} = \sum_{k=0}^{4} 2\binom{4}{k}7^k(-3)^{4-k}x^{8-2k}. \]

The coefficient of \( x^7 \) in the RHS corresponds to the coefficient on the term when \( 8 - 2k = 7 \). But that can never happen and so the coefficient is 0. Similarly,

\[ 2x^3(7 - 3x^2)^4 = \sum_{k=0}^{4} (2x^3)\binom{4}{k}7^k(-3x^2)^{4-k} = \sum_{k=0}^{4} 2\binom{4}{k}7^k(-3)^{4-k}x^{8-2k+3}. \]

Thus, the coefficient of \( x^7 \) is the one of the term corresponding to \( 8 - 2k + 3 = 7 \) which is \( k = 2 \), and so the coefficient is \( 2\binom{4}{2}7^2(-3)^2 \). So, overall, the coefficient of \( x^7 \) in the given expression is

\[ \binom{7}{7}3^7 - \binom{7}{6}3^6 + 2\binom{4}{2}7^2(-3)^2. \]