Problem 1

Suppose we are considering only strings of length 100 that use lower case letters from a to z. How many of these strings are such that no two adjacent positions in the string have the same letter?

**Solution:** Let us “construct” such a string. Here is the process of construction:

- First let’s pick a letter for the first position (from the left) of the string. There are 26 choices for it.
- Now, having fixed the letter for position 1, let’s fix the letter for the second position of the string. It cannot be equal to the first letter, so we have 25 choices for it.
- ...
- For 3 ≤ i ≤ 100, having picked the letters for position 1, . . . , i − 1, let’s pick the letter for the i-th position (from the left) of the string. It cannot be equal to the (i − 1)th position, and so we have 25 choices for it.

Using the generalized product rule, the total number of ways in which the construction can be done is the product of the number of ways for each of the steps, which is $26 \times (25)^{99}$.

Problem 2

Find the coefficient of $x^{101}$ in $\frac{x^3}{2}(1 + x^2)^{50}$?

**Solution:** Finding the coefficient of $x^{101}$ in $\frac{x^3}{2}(1 + x^2)^{50}$ is the same as finding the coefficient of $x^{98}$ in $(1 + x^2)^{50}$ which in turn is $\frac{1}{2}$ times the coefficient of $x^{98}$ in $(1 + x^2)^{50}$. Let us now find the latter. Using the binomial theorem,

$$(1 + x^2)^{50} = \sum_{k=0}^{50} \binom{50}{k} 1^k(x^2)^{50-k},$$
and so the general term in the above sum looks like \( \binom{50}{k} x^{100-2k} \). The terms that will contribute to the coefficient of \( x^{98} \) (or in other words will contain \( x^{98} \)) are those with

\[
100 - 2k = 98 \iff k = 1.
\]

The term with \( k = 1 \) is \( \binom{50}{1} x^{98} \), and so the coefficient of \( x^{98} \) in \( (1 + x^2)^{50} \) is \( \binom{50}{1} \), and so the coefficient of \( x^{101} \) in \( x^3(1 + x^2)^{50} \) is \( \frac{101}{2} = 25 \).

**Problem 3**

Recall that \( \{0, 1\}^n \) is the set of all sequences of length \( n \) where every entry is either 0 or 1. Let us consider the following function \( f : \{0, 1\}^n \to \{0, 1\} : \) for all \((x_1, x_2, \ldots, x_n) \in \{0, 1\}^n\)

\[
f((x_1, x_2, \ldots, x_n)) = x_1 \oplus x_2 \oplus \ldots \oplus x_n,
\]

where \( \oplus \) is the binary XOR operator: \( 1 \oplus 1 = 0, 1 \oplus 0 = 1, 0 \oplus 0 = 0, 1 + 1 + 1 = (1 + 1) + 1 = 0 + 1 = 1 \), etc.

How many points in the domain are mapped to 0 by \( f \), and how many of them are mapped to 1?

**Solution:** By the definition of the XOR operator,

\[
x_1 \oplus x_2 \oplus \ldots \oplus x_n = 0
\]

if and only if the number of ones in the tuple \((x_1, \ldots, x_n)\) is even (if the numbers of ones is odd, then \( x_1 \oplus \ldots \oplus x_n = 1 \). To see why, say we exactly \( 2k \) of the \( x_i \)s are equal to 1 and the rest are zero. We can begin “pairing” the ones to form \( k \) pairs of ones, and since \( 1 \oplus 1 = 0 \) every pair will result in a zero, which means that the XOR of all the \( x_i \)s will be 0. The same idea can be used to show that if the number of ones is \( 2k + 1 \) then we will get a 1 after evaluating the XORs.

Thus, the \( n \)-tuples in the domain \( \{0, 1\}^n \) that map to 0 are those with an even number of ones which is the same as the binary strings of length \( n \) with an even number of ones which is equal to \( 2^{n-1} \). The remaining \( 2^n - 2^{n-1} = 2^{n-1} \) points in the domain (since \( |\{0, 1\}^n| = 2^n \)) all map to 1.

**Problem 4**

Let \( U \) be a universe consisting of \( n \) distinct objects (say \( U = \{u_1, u_2, \ldots, u_n\} \)). We want to partition \( U \) into \( k \) disjoint parts \( A_1, A_2, \ldots, A_k \) such that the size of part \( A_i \) is \( r_i \), i.e. \( |A_i| = r_i \). Of course, since \( A_1, \ldots, A_k \) forms a partition of \( U \), it must be the case that \( r_1 + r_2 + \ldots + r_k = n \). We want to show that the number of ways of doing this is

\[
\frac{n!}{r_1!r_2!\ldots r_k!}.
\]
1. One way of doing this is to first pick \( r_1 \) elements from \( U \) for \( A_1 \), then having picked them, pick \( r_2 \) elements from the remaining elements in \( U \) for \( A_2 \), and so on, until you are only left with \( r_k \) elements and you “put” all of them into \( A_k \). You can convince yourself that the number of ways of doing this process is

\[
\binom{n}{r_1} \binom{n-r_1}{r_2} \binom{n-r_1-r_2}{r_3} \cdots \binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}.
\]

Substitute the value for the binomial coefficients that appear in the above expression and argue that the expression simplifies to

\[
\frac{n!}{r_1! r_2! \cdots r_k!}.
\]

Solution: Substituting the formula for the binomial coefficients we get

\[
\binom{n}{r_1} \binom{n-r_1}{r_2} \binom{n-r_1-r_2}{r_3} \cdots \binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k} = \frac{n!}{r_1!(n-r_1)! r_2!(n-r_1-r_2)! \cdots r_k!(n-r_1-r_2-\cdots-r_{k-1}-r_k)!}.
\]

Notice that a denominator of the 1st term (i.e., \((n - r_1)\)) will cancel the numerator of the second term, a numerator of the second term (i.e., \((n - r_1 - r_2)\)) will cancel the numerator of the third term, and so, on, such that for \( 1 \leq i \leq k - 1 \), a denominator of the \( i \th \) term (in particular, \((n - r_1 - \ldots - r_i)\)) will cancel out the numerator of the \((i + 1)\th \) term, so we get

\[
\frac{n!}{r_1!(n-r_1)! r_2!(n-r_1-r_2)! \cdots r_k!(n-r_1-r_2-\cdots-r_{k-1}-r_k)!} = \frac{n!}{r_1! r_2! \cdots r_{k-1}! r_k!(n-r_1-r_2-\cdots-r_k)!}
\]

where the last equality follows from the fact that \((n-r_1-r_2-\cdots-r_k)! = 0! = 1\) since \( r_1 + r_2 \ldots = r_k = n \).

2. We can also use the bijection method to show this. Let \( A \) be the set of all the ways of partitioning \( U \) into the parts \( A_1, \ldots, A_k \) such that \(|A_i| = r_i\), and let \( B \) be the set of all sequences of length \( n \) whose entries are integers between 1 and \( k \) such that 1 appears \( r_1 \) times in the sequence, 2 appears \( r_2 \) times in the sequence, \( \ldots \), and \( k \) appears \( r_k \) times in the sequence (of course, \( r_1 + r_2 + \ldots + r_k = n \)). Show that there is a bijection \( f : B \to A \).
**Hint:** Consider any sequence \((x_1, x_2, \ldots, x_n) \in \mathcal{B}\). We can use this sequence to define a way of partitioning \(U\) into \(A_1, \ldots, A_k\): if the \(i^{th}\) element of this sequence is \(j\), i.e., \(x_i = j\) (for some number \(1 \leq j \leq k\)) then “put” the \(i^{th}\) object in \(U\) (i.e., \(u_i\)) into the set \(A_j\).

**Solution:** The solution follows by observing that the function described in the hint is indeed a bijection between \(\mathcal{B}\) and \(\mathcal{A}\).

3. Use the previous part along with the BOOKKEEPER rule to conclude that the number of ways of partitioning \(U\) into \(A_1, \ldots, A_k\) such that \(|A_i| = r_i\) is

\[
\frac{n!}{r_1! \cdots r_k!}.
\]

**Solution:** By the BOOKKEEPER rule, the number of sequences in \(\mathcal{B}\) is

\[
\frac{n!}{r_1! \cdots r_k!},
\]

and since there is a bijection between \(\mathcal{A}\) and \(\mathcal{B}\), it follows that

\[
|\mathcal{A}| = |\mathcal{B}| = \frac{n!}{r_1! \cdots r_k!}.
\]

**Problem 5**

There are 100 students in a class. The teacher wants to break them into groups of 5 (so there will be 20 groups in total), and then wants to pick a representative for each group from among the 5 students in the group (i.e., every group will have a representative chosen from the group). In how many ways can the teacher do this?

**Solution:** The process of forming groups and choosing a representative for each group can be done in two steps:

1. First, we split the 100 students into 20 groups of 5. This can be done in \(\frac{100!}{(5!)^{20}}\) ways using the BOOKKEEPER rule (see the previous problem).

2. Having formed the groups, there are \(\binom{5}{1}\) ways of choosing a representative for each group, and there are 20 such groups, and the choices across the groups are independent, and so the total number of ways of choosing the representatives is \(\binom{5}{1}^{20}\).

Using the generalized product rule, the total number of ways of carrying out the process is \(\frac{100!}{(5!)^{20}} \times \binom{5}{1}^{20}\).