Problem 1

There is a bijection between the set of functions with domain \{1, \ldots, 100\} and codomain \{1, 2\}, and the set of 100-tuples \{1, 2\}^{100}. To see why, let \(f : \{1, \ldots, 100\} \to \{1, 2\}\) be any function. Then we can convert it into a 100-tuple in \{1, 2\}^{100} as follows:

\[(f(1), f(2), \ldots, f(100)).\]

Conversely, given a hundred tuple \(s = (s_1, \ldots, s_{100})\) in \{1, 2\}^{100} we can convert it into a function \(f : \{1, \ldots, 100\} \to \{1, 2\}\) by setting

\[f(i) = s_i.\]

Now that we know there is this bijection between tuples and functions, let us try to see how tuples made from monotonic functions look like. Note that a tuple obtained from a monotonic function must look like zero or more 1s followed by zero or more 2s such that the total number of 1s and 2s together is 100. For example, consider the function

\[
\begin{align*}
f(1) &= 1 \\
f(2) &= 1 \\
&\vdots \\
f(50) &= 1 \\
f(51) &= 2 \\
&\vdots \\
f(100) &= 2 \\
\end{align*}
\]

Clearly, \(f\) is monotonic. The tuple obtained from \(f\) looks like 50 ones followed by 50 twos. Thus, we basically want to count the number of tuples of this form: \(x\) ones
followed by $y$ twos, where $x, y \geq 0$, and $x + y = 100$. Using Stars and Bars, the number of such tuples is
\[
\binom{100 + 2 - 1}{2 - 1} = \binom{101}{1} = 101.
\]
This then is the number of monotonic functions.

**Problem 2**

First chose 5 men. $\binom{85}{5}$ ways of doing this. Then choose 5 women. $\binom{15}{5}$ ways of doing that. Thus, the total number of ways of forming a committee is $\binom{85}{5}\binom{15}{5}$.

**Problem 3**

- We can choose either hearts or spades or both. 26 cards in total, and so the answer is $\binom{26}{13}$.

- Let us first decide which two suites to use. There are $\binom{4}{2}$ ways of doing this. We now have to choose 13 cards from the two suits. Since each suite has 13 cards, we need to choose 13 cards from 26 possible cards. We need to ensure that we have at least one card from each of the two suites.

Let $A$ be the set of ways of picking 13 cards from two suites so that we pick at least one card from suite 1. Let $B$ be the set of ways of picking 13 cards from two suites so that we pick at least one card from suite 2. We want to find $|A \cap B|$.

We can use the difference rule:
\[
|A \cap B| = |U| - |\overline{A} \cup \overline{B}|,
\]
where $U$ is the set of all ways of picking 13 cards from 2 suites, and so $|U| = \binom{26}{13}$. Using inclusion-exclusion,
\[
|\overline{A} \cup \overline{B}| = |\overline{A}| + |\overline{B}| - |\overline{A} \cap \overline{B}|.
\]
Note that $|\overline{A}|$ is the number of ways of picking 13 cards from 2 suites so that no card from suite 1 is picked. This is just $\binom{13}{13} = 1$. Similarly, $|\overline{B}| = 1$. Also, note that $|\overline{A} \cap \overline{B}|$ is the number of ways of picking 13 cards from two suites so that no cards are picked from either suite; this is clearly 0. Thus, we have that
\[
|\overline{A} \cup \overline{B}| = 2,
\]
and so
\[
|A \cap B| = \binom{26}{13} - 2.
\]
Thus, the number of ways of first picking two suites, and then picking 13 cards from them so that each suite is represented is
\[
\frac{4}{2} \left(\binom{26}{13} - 2\right)
\]
Problem 4

- Number of even size sets of \( \{1, \ldots, n\} \) is \( 2^{n-1} \), and so the answer in this case is \( 2^8 \).

- Let \( \mathcal{A} \) be the set of all possible subsets \( A \subseteq U \) that satisfy \( |A \cap S| = 4 \) and \( |A \cap T| = 2 \). We can partition \( \mathcal{A} \) into two sets: \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), where \( \mathcal{A}_1 \) is the set of those \( A \in \mathcal{A} \) that contain the number 6, and \( \mathcal{A}_2 \) is the set of all those sets in \( \mathcal{A} \) that don’t contain 6. By sum rule,

\[ |\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|. \]

Let us find the terms on the RHS.

\( |\mathcal{A}_1| \) is the number of sets \( A \subseteq U \) that contain 6 and satisfy \( |A \cap S| = 4 \) and \( |A \cap T| = 2 \). We can construct such a set \( A \) as follows: we already know 6 must be present in \( A \), and since \( |A \cap T| = 2 \), it means that we need to add another element from \( T \) (other than 6) into \( A \). There are \( \binom{3}{1} \) ways of doing this. Then, since \( |A \cap S| = 4 \), and 6 is already in \( A \), we need to pick 3 more elements from \( S \) other than 6 and put them into \( A \). There are \( \binom{5}{3} \) ways of doing this, and so the total number of ways of constructing such sets is \( \binom{3}{1} \binom{5}{3} \).

Similarly, you can show that \( |\mathcal{A}_2| \) is \( \binom{3}{2} \binom{5}{4} \): we need to pick 2 elements from \( T \) (other than 6, since we don’t want the set to contain a 6 this time) to put into the set, and 4 elements from \( S \) (again, other than 6) to put into the set, which gives \( \binom{3}{2} \binom{5}{4} \).

Thus, the overall answer is

\[ |\mathcal{A}| = \binom{3}{1} \binom{5}{3} + \binom{3}{2} \binom{5}{4} \]

Problem 5

- This is basically equivalent to finding solutions to

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 12, \]

where \( x_1, \ldots, x_5 \geq 0 \) are integers, and \( x_i \) represents the number of muffins of type \( i \) that are to be brought. Using stars and bars, the answer is

\[ \binom{12+5-1}{5-1} = \binom{16}{4} \]

- This is basically equivalent to finding solutions to

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 12, \]

where \( x_1, \ldots, x_5 \geq 1 \) are integers, and \( x_i \) represents the number of muffins of type \( i \) that are to be brought. If \( x_1 \) represents the number of banana muffins
bought, then we also have the condition this time that $x_1 \leq 5$. Let $U$ be the set of all possible integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12,$$

with $x_1, \ldots, x_5 \geq 1$, without the $x_1 \geq 5$ constraint.

Let $S$ be the set of all possible integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12,$$

with $x_1, \ldots, x_5 \geq 1$, WITH the $x_1 \leq 5$ constraint. Then using difference method

$$|S| = |U| - |U - S|,$$

here $U - S$ is the set of all integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12,$$

with $x_1, \ldots, x_5 \geq 1$, where $x_1 \leq 5$ is violated, i.e. $x_1 \geq 6$. In other words, $U - S$ is the solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12,$$

with $x_1 \geq 6$ and $x_2, \ldots, x_5 \geq 1$. Using stars and bars,

$$|U - S| = \binom{12 - 10 + 5 - 1}{5 - 1} = \binom{6}{4}.$$

Similarly, $|U| = \binom{11}{4}$ using Stars and Bars method, and so

$$|S| = \binom{11}{4} - \binom{6}{4}.$$

**Problem 6**

Given a binary string of length 25 with 7 ones such that no ones are adjacent, we can use $x_1$ to denote the number of zeros to the left of the first one (from the left), $x_2$ to denote the number of zeros between the first and second ones, and so on, with $x_7$ representing the number of zeros between the 6th and 7th ones. Additionally let $x_8$ denote the number of zeros to the right of the last (7th) one. There are 18 zeros in the string, and so we must have

$$x_1 + \ldots + x_8 = 18,$$

where $x_1, x_8 \geq 0$ (since there could be zero or more 0s to the left of the first one and the right of the last one, furthermore $x_2, \ldots, x_7 \geq 1$, since there is always at least one zero between any two ones. Thus, we see that there is a bijection between strings of the desired form and solutions to the above system of equations. The number of solutions to the above set of equations is $\binom{18 - 6 + 8 - 1}{8 - 1} = \binom{19}{7}$, using the stars and bars method, and so this is also the number of strings of the desired form by the bijection method.
**Problem 7**

Let \( P \) be the set of paths from \((0, 0)\) to \((50, 50)\). Let \( P_1 \) be the set of all paths in \( P \) that pass through \((10, 11)\), and \( P_2 \) be all paths in \( P \) that pass through \((21, 20)\). We want to find the size of \( P_1 \cap P_2 \), i.e. that is all paths neither pass through \((10, 11)\) nor through \((21, 20)\). Using the difference method and De Morgan’s law,

\[
|P_1 \cap P_2| = |P| - |P_1 \cup P_2|.
\]

Note that \(|P| = \binom{100}{50}\) (why?). To find \(|P_1 \cup P_2|\), we use inclusion-exclusion,

\[
|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2|,
\]

where \(|P_1 \cap P_2|\) is all the paths in \( P \) that pass through both \((10, 11)\) and \((21, 20)\).

Let us first find \(|P_1|\). Any path in \( P_1 \) can be broken into two parts: first it goes from \((0, 0)\) to \((10, 11)\), and then from \((10, 11)\) to \((50, 50)\). We can form a full path in \( P_1 \) by combining any path from \((0, 0)\) to \((10, 11)\) with any path from \((10, 11)\) to \((50, 50)\). The number of paths from \((0, 0)\) to \((10, 11)\) is \(\binom{21}{10}\), and the number of paths from \((10, 11)\) to \((50, 50)\) is \(\binom{79}{40}\), and so the number of paths in \( P_1 \) is \(\binom{21}{10}\binom{79}{40}\), and so this is \(|P_1|\).

Similarly, you can compute that \(|P_2| = \binom{43}{29}\).

Let us find \(|P_1 \cap P_2|\). This is the number of paths that first go from \((0, 0)\) to \((10, 11)\), then from \((10, 11)\) to \((21, 20)\), and then from \((21, 20)\) to \((50, 50)\), so there are three parts in every path in \( P_1 \cap P_2 \). The number of choices for the first part is \(\binom{59}{11}\), the number of choices for the second part is \(\binom{20}{10}\), and the number of choices for the third part is \(\binom{59}{29}\), and so

\[
|P_1 \cap P_2| = \binom{21}{10}\binom{20}{10}\binom{59}{29}.
\]

This means that

\[
|P_1 \cup P_2| = |P_1| + |P_2| - |P_1 \cap P_2| = \binom{41}{21}\binom{59}{29} + \binom{21}{10}\binom{79}{40} - \binom{21}{10}\binom{20}{11}\binom{59}{29}.
\]

Then since \(|\bar{P} \cap \bar{P}_2| = |P| - |P_1 \cup P_2|\), we get

\[
|\bar{P} \cap \bar{P}_2| = \binom{100}{50} - \binom{21}{10}\binom{79}{40} - \binom{41}{21}\binom{59}{29} + \binom{21}{10}\binom{20}{11}\binom{59}{29}.
\]

**Problem 8**

Using the multinomial theorem, we have that

\[
((x^7 + 1)^{15} - (4 - z)^{21} + 13y)^{11} = \sum_{a,b,c \geq 0; a + b + c = 11}^{11} \frac{11!}{a!b!c!} (-1)^b 13^c (x^7 + 1)^{15a} (4 - z)^{21b} y^c.
\]

We want to find the coefficients of \(x^7y^8z^9\). The terms that contain this monomial are the ones where \(c = 8\). There are only two such terms

\[
a = 2, b = 1, c = 8
\]
\[ a = 1, b = 2, c = 8. \]

For \( a = 2, b = 1, c = 8 \), the term we get is
\[
\frac{11!}{2!1!8!} (-1)^1 13^8 (x^7 + 1)^{30} (4 - z)^{21} y^8.
\]

To get \( x^7 y^8 z^9 \), we need to get \( x^7 \) from \((x^7 + 1)^{30} \). Using the binomial theorem, the only term that can give us \( x^7 \) in the binomial expansion of \((x^7 + 1)^{30} \) is \((\binom{30}{1}) x^7 \). Thus, the coefficient of \( x^7 y^8 z^9 \) is
\[
\frac{11!}{2!1!8!} (-1)^1 (13)^8 (\binom{30}{1}) (21) (9).
\]

For \( a = 1, b = 2, c = 8 \), the term we get is
\[
\frac{11!}{1!2!8!} (-1)^2 13^8 (x^7 + 1)^{15} (4 - z)^{42} y^8.
\]

To get \( x^7 y^8 z^9 \), we need to get \( x^7 \) from \((x^7 + 1)^{15} \). Using the binomial theorem, the only term that can give us \( x^7 \) in the binomial expansion of \((x^7 + 1)^{15} \) is \((\binom{15}{1}) x^7 \). Similarly, the term that gives \( z^9 \) in \((4 - z)^{42} \) is \((\binom{42}{9}) (-1)^9 (13)^{43} z^9 \). Thus, the coefficient of \( x^7 y^8 z^9 \) is
\[
\frac{11!}{1!2!8!} (-1)^1 (13)^8 (\binom{15}{1}) (42) (9).
\]

Thus, the overall coefficient of \( x^7 y^8 z^9 \) is
\[
\frac{11!}{1!2!8!} (-1)^{11} (13)^8 (\binom{15}{1}) (42) (9) + \frac{11!}{2!1!8!} (-1)^{10} (13)^8 (\binom{30}{1}) (21)
\]

**Problem 9**

We can think of the process of constructing such strings. The process consists of four steps:

1. We first create a string \( x \) of length 40 using 20 \('a\)'s and 20 \('c\)'s. There are \(\binom{40}{20}\) ways of constructing such a string.

2. Then we create a string \( y \) of length 60 containing 20 \('d\)'s, 20 \('e\)'s, and 20 \('f\)'s. The number of ways of doing this using BOOKKEEPER rule is \(\frac{60!}{20!20!20!}\).

3. We now create a string of length 100 consisting of \('a\)'s \('c\)'s \('d\)'s \('e\)'s and \('f\)'s by concatenating \( x \) and \( y \) to get \( xy \) so that the \('a\)'s and \('c\)'s come before the \('d\)'s, \('e\)'s, and \('f\)'s. There is only one way to do this.

4. We will now insert the 20 \('b\)'s into the the length 100 string obtained by completing steps 1-3. Note we can insert the \('b\)'s in between the characters of the string, or before or after the string. It can be verified that there are 101 “slots” that the 20 \('b\)'s can be placed into. Let \( x_1 \) be the number of \('b\)'s placed at the
beginning of the string, let \( x_2 \) be the number of ‘b’s placed between the first and second character, \ldots, let \( x_{100} \) be the number of ‘b’s placed between the 99th and 100th character, and finally let \( x_{101} \) be the number of ‘b’s after the last character. Thus, we have

\[
x_1 + \ldots x_{101} = 20,
\]

where \( x_1, x_2, \ldots, x_{101} \geq 0 \). Thus, the number of ways of placing the ‘b’s is the number of solutions to the above equation, which using stars and bars is

\[
\binom{20+101-1}{101-1} = \binom{120}{100}.
\]

It can be observed that each way of doing this process results in a unique string of the desired form, and also every string of the desired form can be obtained via this process. Thus, the number of strings of the desired form is equal to the number of ways of doing the above process, which using the product rule is

\[
\frac{\binom{40}{20}}{20!} \frac{60!}{120!} \frac{120}{100}.
\]

**Problem 10**

- (a) If there is a 011 starting at position 3, then basically positions 3, 4, 5 are fixed, but the rest \( n - 3 \) positions can be filled with 0 and 1 in an independent manner, thus the number of strings is \( r = 2^{n-3} \).

- (b) If there are two 011s in the string, one starting at position \( i \) and one starting at position \( j \), then basically 6 out of \( n \) positions are already fixed and the rest \( n - 6 \) positions can be filled in whatever way we want using 0s and 1s. Thus, the number of such strings is \( s = 2^{n-6} \).

- (c) We want to choose \( i \) and \( j \) in \( \{0, 1, \ldots, n - 3\} \) such that \( i < j \), and furthermore \( j \) is not \( i + 1 \) or \( i + 2 \) (so that the 011 starting at \( i \) does not intersect with the 011 starting at positions \( j \)). Thus, \( j - i \geq 3 \), i.e. \( i \) and \( j \) have at least 2 numbers between them. Let us think of \( \{0, 1, \ldots, n - 3\} \) as \( n - 2 \) seats, and choosing \( i < j \) that satisfies the \( j - i \geq 3 \) is like seating two students \( i \) and \( j \) in these seats so that the are separated by at least two seats, i.e. have at least two seats between them. Let \( x_1 \) be the number of seats to the left of \( i \), let \( x_2 \) be the number of seats between \( i \) and \( j \), and let \( x_3 \) be the number of seats to the right of \( j \). Clearly,

\[
x_1 + x_2 + x_3 = n - 4,
\]

because \( n - 4 \) seats are unoccupied. Also, \( x_1, x_3 \geq 0 \) but \( x_2 \geq 2 \) since we want there to be two seats between \( i \) and \( j \). Thus, the number of ways of choosing \( i \) and \( j \) from \( \{0, \ldots, n-3\} \) so that \( j - i \geq 3 \) is the same as the number of solutions to the above equation with the above constraints. Thus number of solutions is

\[
\binom{n - 4 - 2 + 3 - 1}{3 - 1} = \binom{n - 4}{2}
\]

and so \( t = \binom{n-4}{2} \).
(d) We want to find all strings of length \( n = 9 \) that contain at least one 011. Note that strings of length 9 can contain at most 3 011s. This is equivalent to finding
\[
|A_0 \cup \ldots \cup A_8|.
\]
Using inclusion-exclusion this expression will have 9 levels. However, notice that level \( \ell \) corresponds to terms of the form \(|A_{i_1} \cap \ldots \cap A_{i_\ell}|\), i.e. all strings which have \( \ell \) 011s, one starting at position \( i_1 \), one starting at \( i_2 \), \ldots, and one starting at \( i_\ell \). But if \( \ell > 3 \), then this is zero because we can have at most 3 011s in a string of length 9. Thus, there are going to be only 3 levels. Furthermore, the only way there can be three 011s, is if the first starts at 0 then next starts at 3 and the third starts at 6. Thus, the only non-zero term at level 3 is \(|A_0 \cap A_3 \cap A_6|\), all others are zero. Also note that
\[
|A_0 \cap A_3 \cap A_6| = 1,
\]
since there is only one such string. Also, the sign at level 3 is +.

At level two we will have terms like \(|A_i \cap A_j|\). If \( i \) and \( j \) are too close to each other, i.e. \( j - i < 3 \), or if \( i \) or \( j \) is greater than \( n - 3 \) then the number of strings which contain a 011 starting at \( i \) and another starting at \( j \) is 0. From part (c), we know that the number of \( i < j \) that will give us two non-overlapping 011s at positions \( i \) and \( j \) is \( t \). Furthermore each term that is non-zero at level 2 is equal in value to \( s \) by part (b), and the sign of all terms at level 2 is \((-1)^{2-1} = -1\), i.e negative. Thus, the sum of all terms at level 2 with right sign is \(-ts\).

At level 1, the expression is \( \sum_{i=0}^{8} |A_i| \). Obviously, \( A_8, A_7 \) are empty because there is no way a 011 can start at position 7 or 8. Thus, the only non-zero terms are \(|A_0|, \ldots, |A_6|\), i.e. 7 terms. Each term in equal to \( r = 2^{9-3} \), since these are strings three of whose positions are fixed (they are 011) while the other \( n - 3 = 9 - 3 \) are positions can be filled in in any way using 0 and 1. Thus,
\[
\sum_{i=0}^{8} |A_i| = 7r.
\]

Thus, using inclusion exclusion,
\[
|A_0 \cup \ldots \cup A_8| = 7r - ts + 1
\]