CS 344

LECTURE 15

LINEAR PROGRAMMING
LINEAR PROGRAMMING

- bipartite matching
- duality
- zero-sum games
BIPARTITE GRAPH

A bipartite graph is one where we can partition the vertices into two sets $V_1$ and $V_2$, such that there are no edges between vertices in the same group.
Let's say we have guys and girls and we know which pairs like each other:

Is there a choice of couples so everyone is paired up?

(a perfect matching in graph theory)
We can reduce this to a problem we've seen before by adding a source and sink:
If each edge has weight one, then we want the max flow of the graph!
Recall that the Ford-Fulkerson algorithm found the max flow of a graph.

We showed maximum flow $\leq$ minimum cut.

And from the residual graph, we constructed a cut that had a value equal to our flow.

Therefore, the flow we found was optimal!
In general, a LP maximization problem can be translated into a "dual" minimization problem.

\[
\text{max flow} \iff \text{min cut}
\]
Recall the chocolate shop:

$$\max x_1 + 6x_2$$

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1, x_2 \geq 0$$
\[
\begin{align*}
\text{max } x_1 + 6x_2 \\
x_1 &\leq 200 \\
x_2 &\leq 300 \\
x_1 + x_2 &\leq 400 \\
x_1, x_2 &\geq 0
\end{align*}
\]
If we add the first inequality and six times the second:

\[ x_1 + 6x_2 \leq 2000 \]

So we can never get a profit above 2000.
If we multiply the three inequalities by 0, 5, and 1 instead:

\[ x_1 + 6x_2 \leq 1900 \]

This proves our solution is optimal!
<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

Note that each $y_i$ must be nonnegative.
This gives us the bound:

\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400\]
To get the LHS to look like our objective function, we could set

\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400\]

\[y_1 + y_3 = 1\]
\[y_2 + y_3 = 6\]
\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400\]

But as a bound, we really only need:

\[
y_1 + y_3 \geq 1
\]

\[
y_2 + y_3 \geq 6
\]
Finally, we get our bound:

\[ x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3 \quad \text{if} \quad y_1, y_2, y_3 \geq 0 \]

\[ y_1 + y_3 \geq 1 \]

\[ y_2 + y_3 \geq 6 \]
For example, \((y_1, y_2, y_3) = (5, 3, 6)\) gives a loose upper bound of 4300.

To find the tightest upper bound, though, we must minimize the RHS: \(200y_1 + 300y_2 + 400y_3\)
Thus finding the best upper bound on the original gives us a new LP:

\[
\begin{align*}
\text{min} & \quad 200y_1 + 300y_2 + 400y_3 \\
y_1, y_2, y_3 & \geq 0 \\
y_1 + y_3 & \geq 1 \\
y_2 + y_3 & \geq 6
\end{align*}
\]
This *dual* LP gives upper bounds on the original *primal* LP.

Finding values that makes them equal means they must be optimal:

- \((x_1, x_2) = (100, 300)\)
- \((y_1, y_2, y_3) = (0, 5, 1)\)
This duality gap is zero
DUALITY THEOREM

If a linear program has a bounded optimum, then so does its dual, and they're equal
Primal LP:

\[ \max c^T x \]
\[ Ax \leq b \]
\[ x \geq 0 \]

Dual LP:

\[ \min y^T b \]
\[ y^T A \geq c^T \]
\[ y \geq 0 \]
We may have inequalities \((I)\) and equalities \((E)\):

Primal LP:

\[
\begin{align*}
\text{max} & \quad c_1x_1 + \cdots + c_n x_n \\
 a_{i1}x_1 + \cdots + a_{in}x_n & \leq b_i \quad \text{for } i \in I \\
a_{i1}x_1 + \cdots + a_{in}x_n & = b_i \quad \text{for } i \in E \\
x_j & \geq 0 \quad \text{for } j \in N
\end{align*}
\]

Dual LP:

\[
\begin{align*}
\text{min} & \quad b_1y_1 + \cdots + b_m y_m \\
a_{1j}y_1 + \cdots + a_{mj}y_m & \geq c_j \quad \text{for } j \in N \\
a_{1j}y_1 + \cdots + a_{mj}y_m & = c_j \quad \text{for } j \not\in N \\
y_i & \geq 0 \quad \text{for } i \in I
\end{align*}
\]
Primal LP

\[
\begin{align*}
\text{max} & \quad x_1 + 6x_2 \\
x_1 & \leq 200 \\
x_2 & \leq 300 \\
x_1 + x_2 & \leq 400 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{min} & \quad 200y_1 + 300y_2 + 400y_3 \\
y_1, y_2, y_3 & \geq 0 \\
y_1 + y_3 & \geq 1 \\
y_2 + y_3 & \geq 6
\end{align*}
\]
We can view the constraints of the primal LP as the rows of $A$, and the dual as the columns:

\[
\begin{align*}
  x_1 & \leq 200 \\
  x_2 & \leq 300 \\
  x_1 + x_2 & \leq 400
\end{align*}
\implies
\begin{pmatrix}
  1 & 0 \\
  0 & 1 \\
  1 & 1
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\leq
\begin{pmatrix}
  200 \\
  300 \\
  400
\end{pmatrix}
\]
ZERO-SUM GAMES

If we have a cake, and you take a piece, you gain a piece of cake, but there's one fewer piece for everyone else.

$$\sum \text{gains} - \sum \text{losses} = 0$$
ROCK, PAPER, SCISSORS
Then the payoff to Row gives this matrix:

\[ G = \begin{array}{c|ccc}
\text{Column} \\
\hline
r & 0 & -1 & 1 \\
p & 1 & 0 & -1 \\
s & -1 & 1 & 0 \\
\end{array} \]
Then each can pick a probabilistic strategy:

- Row strategy \( x = (x_1, x_2, x_3) \)
- Col strategy \( y = (y_1, y_2, y_3) \)
\[
\sum_{i,j} G_{ij} \cdot \text{Prob}[\text{Row plays } i, \text{ Column plays } j] = \sum_{i,j} G_{ij} x_i y_j
\]

Note that Row wants to maximize this, and Col wants to minimize it.
If Row uses a uniformly random strategy and Col plays $r$:

$$\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = 0$$
In general, for any strategy Col chooses:

\[
\sum_{i,j} G_{ij} x_i y_j = \sum_{i,j} G_{ij} \cdot \frac{1}{3} y_j = \sum_{j} y_j \left( \sum_{i} \frac{1}{3} G_{ij} \right) = \sum_{j} y_j \cdot 0 = 0
\]
Row playing randomly forces 0 payoff for Col
Col playing randomly forces 0 payoff for Row
Suppose we require each player to announce their strategy first:

- First Row announces, then Col
- First Col announces, then Row

Seems like first to announce would be at a disadvantage...
Consider an election and platform strategies...

- economy
- society
- morality
- taxes
\[ G = \begin{array}{cc}
& m & t \\
e & 3 & -1 \\
s & -2 & 1 \\
\end{array} \]

values are votes lost by column
\[ G = \begin{array}{cc}
  e & m & t \\
  s & 3 & -1 \\
  & -2 & 1 \\
\end{array} \]

So Col picks \( t \)

For \( x = (1/2, 1/2) \), we have:

- \( m \) yields a loss of \( 1/2 \)
- \( t \) yields a loss of \( 0 \)

Therefore, Col picks \( t \).
Given a \((x_1, x_2)\), Col picks a pure strategy – \(m\) or \(t\)

If Row announces, Col will choose

\[
\min\{3x_1 - 2x_2, -x_1 + x_2\}
\]

Row should pick \((x_1, x_2)\) that maximizes

\[
\min\{3x_1 - 2x_2, -x_1 + x_2\}
\]
\[ z = \min\{3x_1 - 2x_2, -x_1 + x_2\} \]

can be rewritten as

\[ \max z \]

\[ z \leq 3x_1 - 2x_2 \]

\[ z \leq -x_1 + x_2 \]
Row wants to maximize $z$:

$$\text{max } z$$

$$-3x_1 + 2x_2 + z \leq 0$$
$$x_1 - x_2 + z \leq 0$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$
If Col goes first, Col wants to minimize
\[ \max\{3y_1 - y_2, -2y_1 + y_2\} \]

\[
G \quad = \quad \begin{array}{c|cc}
  & m & t \\
\hline
  e & 3 & -1 \\
  s & -2 & 1 \\
\end{array}
\]
Col wants to minimize $w$:

$$
\begin{align*}
\text{min } w \\
-3y_1 + y_2 + w &\geq 0 \\
2y_1 - y_2 + w &\geq 0 \\
y_1 + y_2 &= 1 \\
y_1, y_2 &\geq 0
\end{align*}
$$
Row wants to maximize $z$:

$$\begin{align*}
\text{max } z \\
-3x_1 + 2x_2 + z &\leq 0 \\
x_1 - x_2 + z &\leq 0 \\
x_1 + x_2 &= 1 \\
x_1, x_2 &\geq 0
\end{align*}$$

Col wants to minimize $w$:

$$\begin{align*}
\text{min } w \\
-3y_1 + y_2 + w &\geq 0 \\
2y_1 - y_2 + w &\geq 0 \\
y_1 + y_2 &= 1 \\
y_1, y_2 &\geq 0
\end{align*}$$
These two LPs are duals to each other, so have an optimum, $V$

$V$ is known as the *value* of the game
In this case, the optimum is at

- Row: \((3/7, 4/7)\)
- Col: \((2/7, 5/7)\)

with value \(V = 1/7\)
This can be generalized to other games and is known as the *Min-Max Theorem* in game theory:

\[
\max_x \min_y \sum_{i,j} G_{ij} x_i y_j = \min_y \max_x \sum_{i,j} G_{ij} x_i y_j
\]